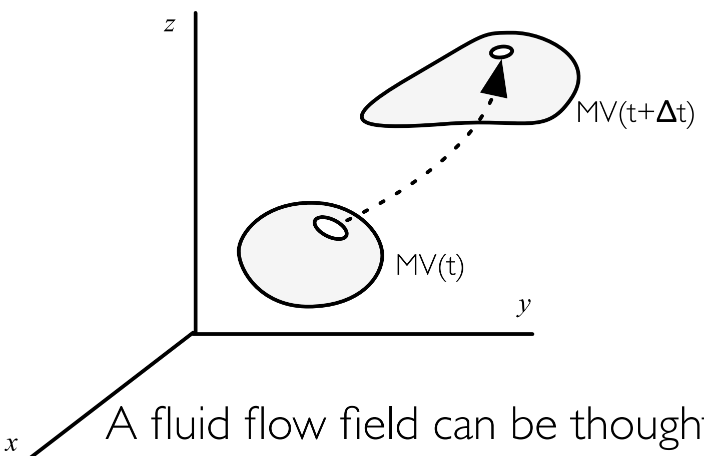
Mathematical Description of Physical Phenomena

Chapter 03

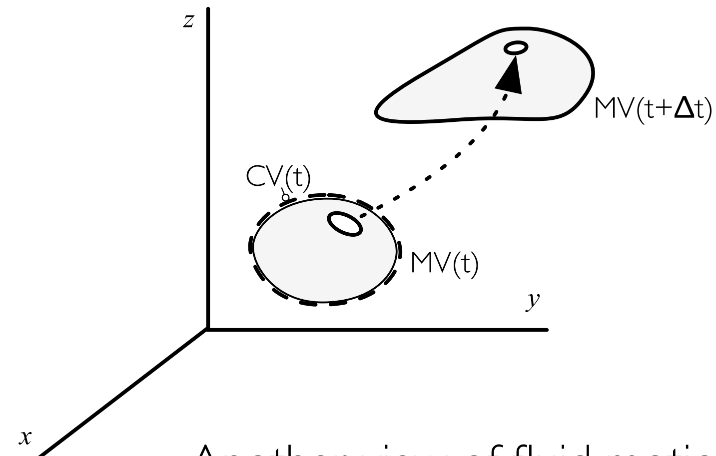
Conservation Equations

Physical Principles	→	Mathematical Equations
Mass conservation Momentum Conservation Energy Conservation Specie Conservation		Continuity Equation Momentum Equation Energy Equation Specie Equation

Lagrangian vs Eulerian



A fluid flow field can be thought of as being comprised of a large number of finite sized fluid particles which have mass, momentum, internal energy, and other properties. Mathematical laws can then be written for each fluid particle. This is the Lagrangian description of fluid motion.



Another view of fluid motion is the Eulerian description. In the Eulerian description of fluid motion, we consider how flow properties change at a fluid element that is fixed in space and time (x,y,z,t), rather than following individual fluid particles.

Governing equations can be derived using either framework and converted to the other form.

Reynolds Transport Theorem

$$\phi(x,y,z,t)$$

Material or substantial derivative

Local time derivative

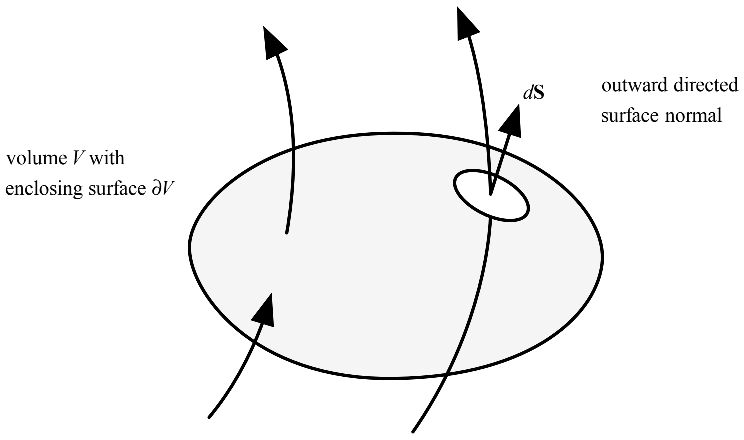
$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{\partial \phi}{\partial t} + \mathbf{v}\cdot\nabla\phi$$

Convective derivative

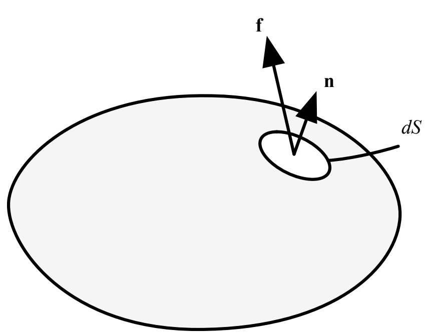
Rate of change for a moving particle

Rate of change at a fixed point



$$\frac{d}{dt}\iint_{\Omega_t}\phi(\mathbf{r},t)\,d\Omega = \frac{\partial}{\partial t}\iint_{\Omega=\Omega_t}\phi(\mathbf{r},t)\,d\Omega + \oint_{\partial\Omega}\phi(\mathbf{r},t)(\mathbf{v}-\mathbf{w})\cdot d\mathbf{S}$$

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \dot{\Phi}_{out} - \dot{\Phi}_{in}$$



Transport Theorem

For any global system property B and property per unit mass b=B/m

$$\left(\frac{dB}{dt}\right)_{MV} = \frac{d}{dt}\left(\int_{V(t)} b\rho\, dV\right) + \int_{S(t)} b\rho\mathbf{v}_r \cdot \mathbf{n}\, dS$$

$$\frac{d}{dt}\left(\int_V b\rho\, dV\right) = \int_V \frac{\partial}{\partial t}(b\rho)\, dV$$

$$\left(\frac{dB}{dt}\right)_{MV} = \int_V \frac{\partial}{\partial t}(b\rho)\, dV + \int_S b\rho\mathbf{v} \cdot \mathbf{n}\, dS$$

$$\left(\frac{dB}{dt}\right)_{MV} = \int_V \left[\frac{\partial}{\partial t}(\rho b) + \nabla \cdot (\rho\mathbf{v}b)\right] dV$$

$$\left(\frac{dB}{dt}\right)_{MV} = \int_V \left[\frac{D}{Dt}(\rho b) + \rho b \nabla \cdot \mathbf{v}\right] dV$$

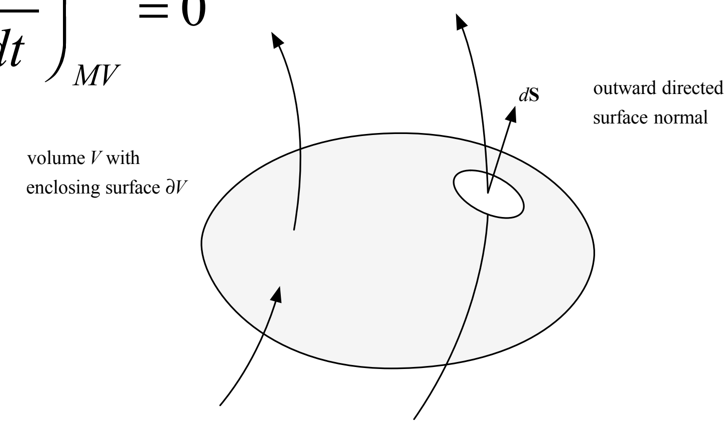
At t=0 the control volume occupy the same space as the control mass.
Physical laws are derived for control mass

Conservation of mass

System property $B = m$

Property per unit mass $b = \frac{m}{m} = 1$

$$\left(\frac{dm}{dt}\right)_{MV} = 0$$

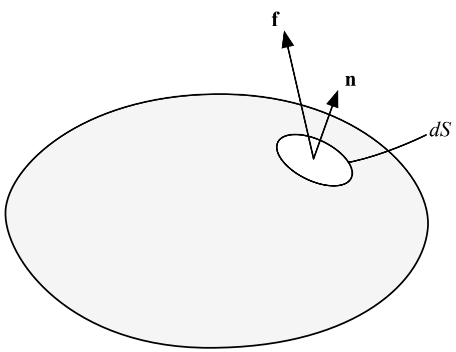


$$\int_V \left[\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v}\right] dV = 0$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot \left[\rho \mathbf{v}\right] = 0 \qquad \nabla \cdot \mathbf{v} = 0$$

Second Law of Motion

$$\left(\frac{d\left(m\mathbf{v}\right)}{dt}\right)_{MV} = \left(\int_V \mathbf{f}\, dV\right)_{MV}$$



System property $$B = m\mathbf{v}$$

Property per unit mass $$b = \frac{m\mathbf{v}}{m} = \mathbf{v}$$

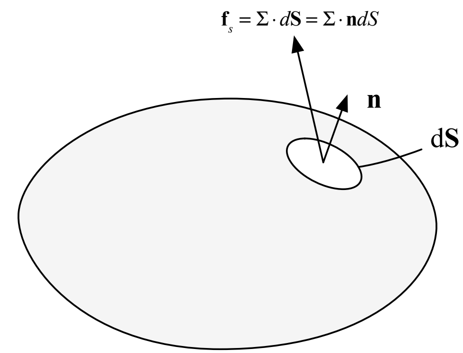
$$\int_V \left[\frac{\partial}{\partial t}\left[\rho\mathbf{v}\right] + \nabla\cdot\left\{\rho\mathbf{v}\mathbf{v}\right\} - \mathbf{f}\right] dV = 0$$

$$\frac{\partial}{\partial t}\left[\rho\mathbf{v}\right] + \nabla\cdot\left\{\rho\mathbf{v}\mathbf{v}\right\} = \mathbf{f}$$

$$\mathbf{f} = \mathbf{f}_s + \mathbf{f}_b$$ Volume forces

Surface forces

$$\boldsymbol{\Sigma} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \overbrace{\Sigma_{xx} + p}^{\tau_{xx}} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \overbrace{\Sigma_{yy} + p}^{\tau_{yy}} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \overbrace{\Sigma_{zz} + p}^{\tau_{zz}} \end{pmatrix} = -p\mathbf{I} + \boldsymbol{\tau}$$



$$p = -\frac{1}{3}\left(\Sigma_{xx} + \Sigma_{yy} + \Sigma_{zz}\right)$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{zx} & \Sigma_{zy} & \Sigma_{zz} \end{pmatrix}$$

$$\int_V \mathbf{f}_s \, dV = \int_S \boldsymbol{\Sigma} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \boldsymbol{\Sigma} \, dV \Rightarrow \mathbf{f}_s = \left[\nabla \cdot \boldsymbol{\Sigma}\right] = -\nabla p + \left[\nabla \cdot \boldsymbol{\tau}\right]$$

$$\frac{\partial}{\partial t}\left[\rho \mathbf{v}\right] + \nabla \cdot \left\{\rho \mathbf{v}\mathbf{v}\right\} = -\nabla p + \left[\nabla \cdot \boldsymbol{\tau}\right] + \mathbf{f}_b \qquad \boldsymbol{\tau} = \mu\left\{\nabla \mathbf{v} + \left(\nabla \mathbf{v}\right)^{\mathrm{T}}\right\} + \lambda\left(\nabla \cdot \mathbf{v}\right)\mathbf{I}$$

$$\boldsymbol{\tau} = \begin{bmatrix} 2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\mathbf{v} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{v} & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{v} \end{bmatrix}$$

$$\left[\nabla\cdot\boldsymbol{\tau}\right] = \nabla\cdot\left[\mu\left(\nabla\mathbf{v} + \left(\nabla\mathbf{v}\right)^{\mathrm{T}}\right)\right] + \nabla\left(\lambda\nabla\cdot\mathbf{v}\right)$$

$$= \begin{bmatrix} \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\mathbf{v}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] \\ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{v}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] \\ \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{v}\right] \end{bmatrix}$$

Momentum Equation

$$\frac{\partial}{\partial t}\left[\rho \mathbf{v}\right]+\nabla\cdot\left\{\rho\mathbf{v}\mathbf{v}\right\}=-\nabla p+\nabla\cdot\left\{\mu\left[\nabla\mathbf{v}+\left(\nabla\mathbf{v}\right)^{\mathrm{T}}\right]\right\}+\nabla\left(\lambda\nabla\cdot\mathbf{v}\right)+\mathbf{f}_b$$

$$\frac{\partial}{\partial t}\left[\rho \mathbf{v}\right]+\nabla\cdot\left\{\rho\mathbf{v}\mathbf{v}\right\}=\nabla\cdot\left\{\mu\nabla\mathbf{v}\right\}-\nabla p+\underbrace{\nabla\cdot\left\{\mu\left(\nabla\mathbf{v}\right)^{\mathrm{T}}\right\}+\nabla\left(\lambda\nabla\cdot\mathbf{v}\right)+\mathbf{f}_b}_{\mathbf{Q}^{\mathrm{v}}}$$

$$\frac{\partial}{\partial t}\left[\rho \mathbf{v}\right]+\nabla\cdot\left\{\rho\mathbf{v}\mathbf{v}\right\}=\nabla\cdot\left\{\mu\nabla\mathbf{v}\right\}-\nabla p+\mathbf{Q}^{\mathrm{v}}$$

$$\frac{\partial}{\partial t}\left[\rho \mathbf{v}\right]+\nabla\cdot\left\{\rho\mathbf{v}\mathbf{v}\right\}=-\nabla p+\nabla\cdot\left\{\mu\left[\nabla\mathbf{v}+\left(\nabla\mathbf{v}\right)^{\mathrm{T}}\right]\right\}+\mathbf{f}_b$$

Momentum Equation

$$\mu \frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x}\right] + \mu \frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \mu \frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$= \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial yx} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial zx}\right]$$

viscosity is constant

$$= \mu\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial^2 u}{\partial x^2} + + \frac{\partial^2 v}{\partial yx} + + \frac{\partial^2 w}{\partial zx}\right]$$

$$= \mu\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$$

$$\frac{\partial}{\partial t}\left[\rho \mathbf{v}\right] + \nabla \cdot \left\{\rho \mathbf{v}\mathbf{v}\right\} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f}_b$$

$$\frac{\partial}{\partial t}\left[\rho \mathbf{v}\right] + \nabla \cdot \left\{\rho \mathbf{v}\mathbf{v}\right\} = -\nabla p + \mathbf{f}_b$$

invscid

Conservation of Energy

$$\left(\frac{dE}{dt}\right)_{MV} = \dot{Q} - \dot{W}$$

Rate of change of the energy of the system is equal to the rate of transfer of energy

System property $\quad B = E$

Property per unit mass $\quad B = E \Rightarrow b = \frac{dE}{dm} = \hat{u} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} = e$

$$\frac{dE}{dt} = \iint_{\Omega=\Omega_t} \frac{\partial(\rho e)}{\partial t} d\Omega + \oint_{\partial\Omega} \rho e(\mathbf{v} - \mathbf{w})\cdot d\mathbf{S} = \dot{Q} - \dot{W}$$

$$\frac{dE}{dt} = \iint_{\Omega=\Omega_t} \left\{\frac{\partial(\rho e)}{\partial t} + \nabla\cdot\left(\rho e[\mathbf{v} - \mathbf{w}]\right)\right\} d\Omega = \dot{Q} - \dot{W}$$

where

$$e = C_v T + \frac{v^2}{2} + \frac{p}{\rho} - \mathbf{g}\cdot\mathbf{r}$$

Conservation of Energy

$$\frac{dE}{dt} = \iint_{\Omega=\Omega_t} \left\{ \frac{\partial(\rho e)}{\partial t} + \nabla\cdot\left(\rho e[\mathbf{v}-\mathbf{w}]\right) \right\} d\Omega$$

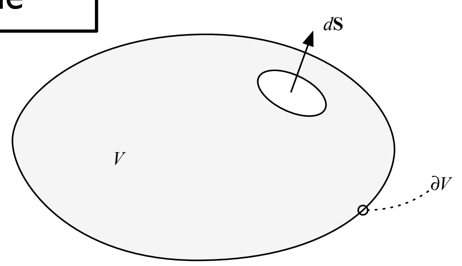
$$= \underbrace{\oint_{\partial\Omega} k\nabla T \cdot d\mathbf{S}}_{\text{Flux of heat through the surface}} + \underbrace{\oint_{\partial\Omega} \mathbf{v}\cdot p\, d\mathbf{S}}_{\text{mechanical contribution of heat arising from compression and viscous dissipation}} + \underbrace{\iint_{\Omega=\Omega_t} \left(\Phi + \rho\mathbf{v}\cdot\mathbf{g}\right) d\Omega}_{\textit{Internal heat generation and body forces}}$$

$$= \iint_{\Omega=\Omega_t} \left\{ \nabla\cdot(k\nabla T) + \nabla\cdot(\mathbf{v}p) + \Phi + \rho\mathbf{v}\cdot\mathbf{g} \right\} d\Omega$$

$$\frac{\partial(\rho e)}{\partial t} + \nabla\cdot\left(\rho e[\mathbf{v}-\mathbf{w}]\right) = \nabla\cdot(k\nabla T) + \nabla\cdot(\sigma\cdot\mathbf{v}) + \Phi + \rho\mathbf{v}\cdot\mathbf{g}$$

Generalized Form

Change of of ϕ over time Δt within the material volume (MV)	=	Surface flux of ϕ over time Δt across the control volume	+	Source/sink of ϕ over time Δt within control volume
Term I		Term II		Term III



$$\text{Term I} = \frac{d}{dt}\left(\int_{MV} (\rho\phi)\, dV \right) = \int_V \left[\frac{\partial}{\partial t}(\rho\phi) + \nabla\cdot(\rho\mathbf{v}\phi) \right] dV$$

$$\text{Term II} = -\int_S \mathbf{J}^{\phi}_{diffusion} \cdot \mathbf{n}\, dS = -\int_V \nabla\cdot\mathbf{J}^{\phi}_{diffusion}\, dV = \int_V \nabla\cdot\left(\Gamma^{\phi}\nabla\phi\right) dV$$

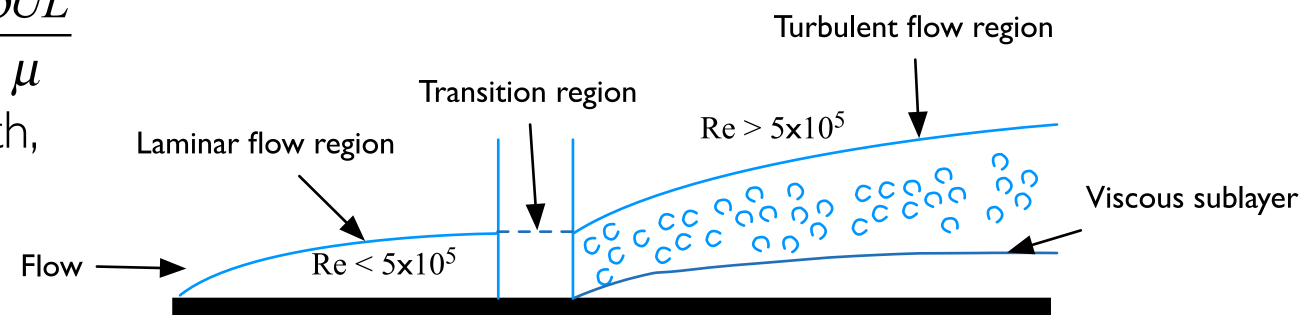
$$\text{Term III} = \int_V Q^{\phi}\, dV$$

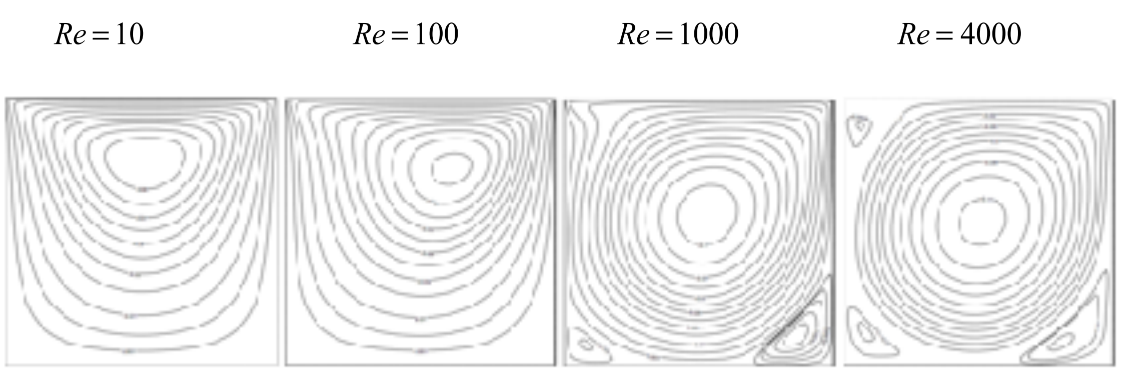
$$\underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{\text{unsteady term}} + \underbrace{\nabla\cdot(\rho\mathbf{v}\phi)}_{\text{convection term}} = \underbrace{\nabla\cdot\left(\Gamma^{\phi}\nabla\phi\right)}_{\text{diffusion term}} + \underbrace{Q^{\phi}}_{\text{source term}}$$

Characterizing Flows

Reynolds Number

- The Reynolds number Re is defined as: $$Re = \frac{\rho U L}{\mu}$$
- Here L is a characteristic length, and V is the velocity.
- It is a measure of the ratio between inertial forces and viscous forces.
- If Re >> 1 the flow is dominated by inertia.
- If Re << 1 the flow is dominated by viscous effects (Creeping flow)
  - Microfluids
  - Flows in narrow passages



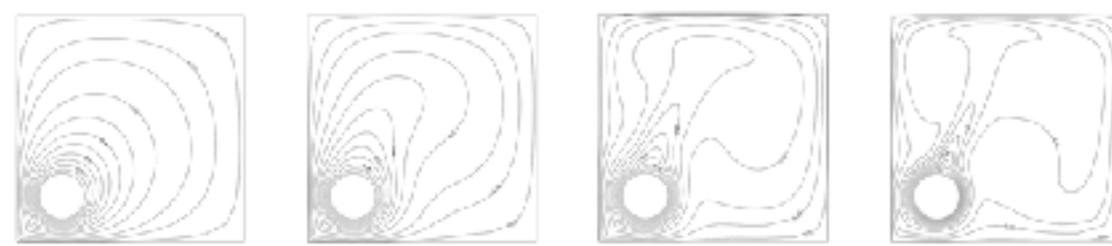


Grashof Number

- When Gr >> 1, the viscous force is negligible compared to the buoyancy and inertial forces. When buoyant forces overcome the viscous forces, the flow starts a transition to the turbulent regime.
- For a flat plate in vertical orientation, this transition occurs around Gr = 10e+9.
- The Grashof number is analogous to the Reynolds number in forced convection.

$$Gr = \frac{g\beta \Delta T L^3}{\nu^2} = \frac{Buoyancy\ \ force}{Viscous\ \ force}$$

$Gr = 1.43 \times 10^3$ $Gr = 1.43 \times 10^4$ $Gr = 1.43 \times 10^5$ $Gr = 1.43 \times 10^6$

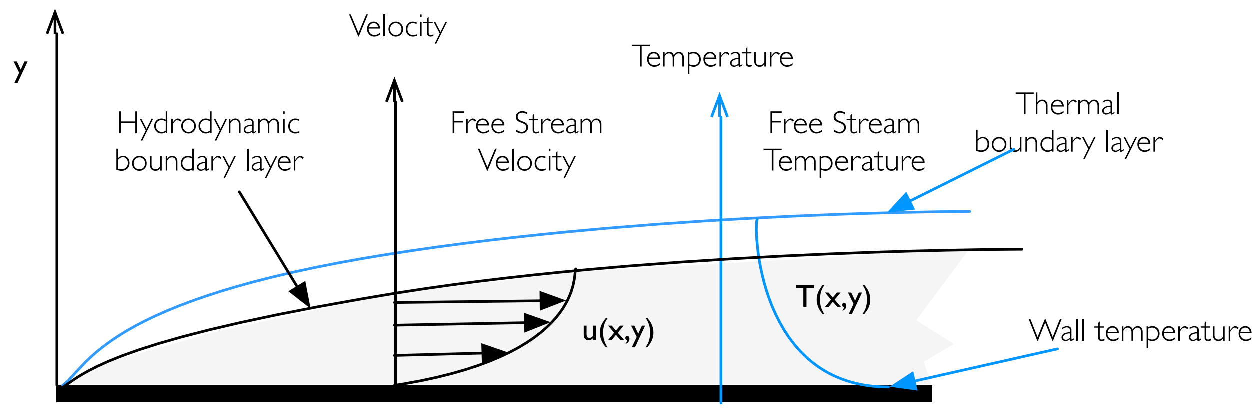


Prandtl Number

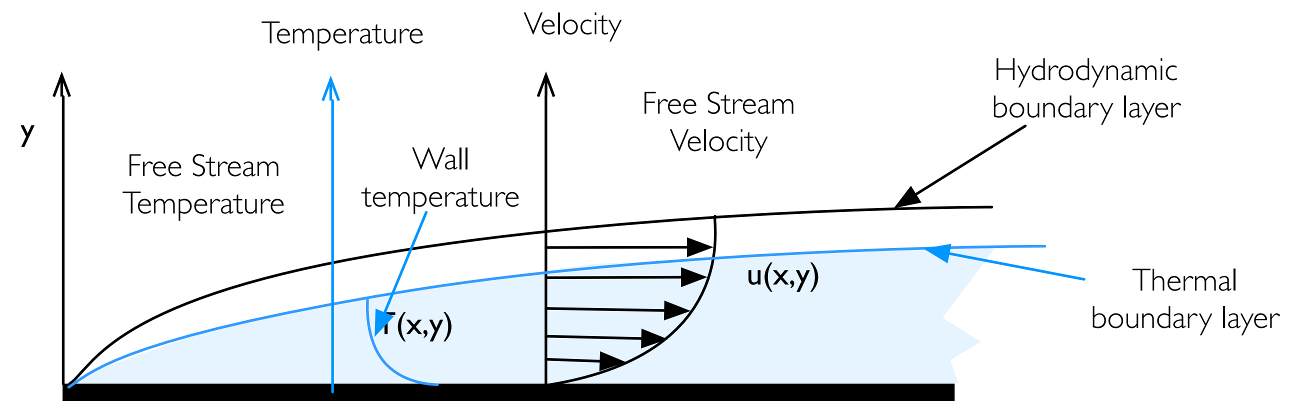
$$Pr = \frac{\mu c_p}{k} == \frac{viscous\ diffusion\ rate}{thermal\ diffusion\ rate} = \frac{\upsilon}{\alpha}$$

- In heat transfer problems, the Prandtl number controls the relative thickness of the momentum and thermal boundary layers.

- When Pr is small, it means that the heat diffuses very quickly compared to the velocity (momentum). This means that for liquid metals the thickness of the thermal boundary layer is much bigger than the velocity boundary layer.



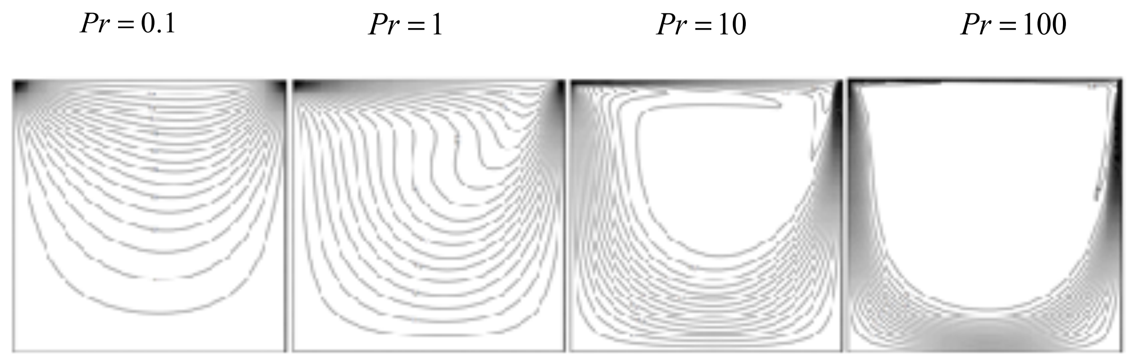
$Pr < 1$



$Pr > 1$

Prandtl Number

- Typical values for Pr are:
  - around 0.7-0.8 for air and many other gases,
  - around 0.16-0.7 for mixtures of noble gases or noble gases with hydrogen
  - around 7 for water (At 20 degrees Celsius)
  - between 100 and 40,000 for engine oil,
  - between 4 and 5 for R-12 refrigerant
  - around 0.015 for mercury.



$(Re = 100)$

Isotherms at increasing values of Prandtl number for driven flow in a square cavity

Peclet Number

The Péclet number is defined as the ratio of the advective transport rate of a physical quantity to its diffusive transport rate. For the case of heat transfer, the Péclet number is given by

$$Pe = \frac{\rho U L c_p}{k} = \frac{UL}{\alpha} = Re * Pr$$

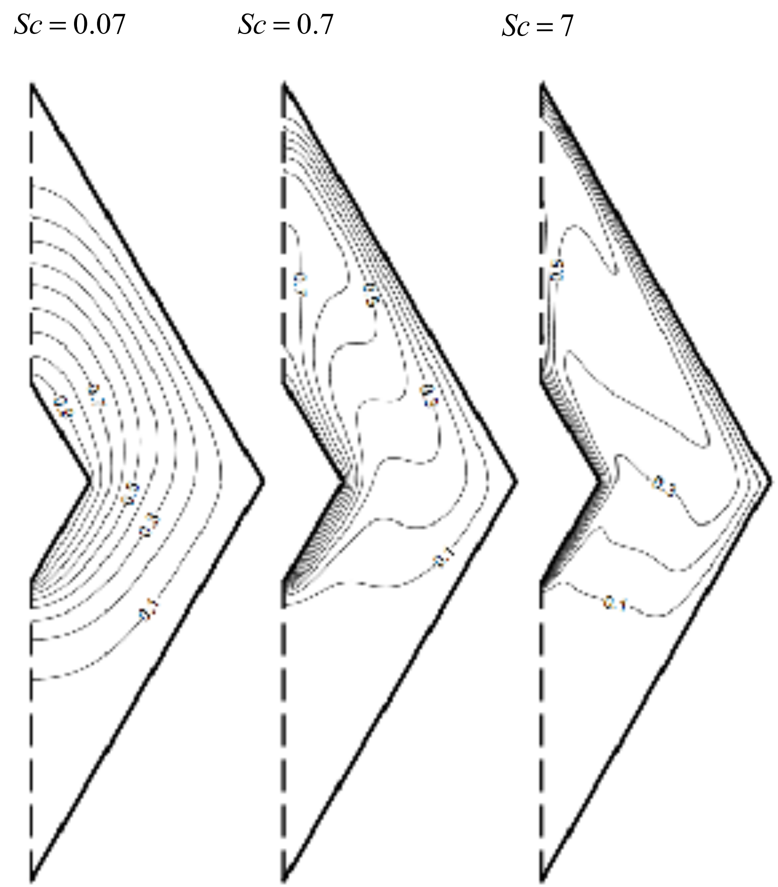
$Pe = 1$ $Pe = 10$ $Pe = 100$ $Pe = 1000$

Isotherms at increasing values of Péclet number for fluid flow over a flat plate maintained at a hot uniform temperature.

Schmidt Number

$$Sc = \frac{\nu}{D}$$

The Schmidt number in mass transfer is the counterpart of the Prandtl number in heat transfer. It represents the ratio of the momentum diffusivity to mass diffusivity



Iso-concentrations at increasing values of Schmidt number (other parameters held fixed) for natural convection mass transfer in the annulus between concentric horizontal cylinders of rhombic cross sections with larger solute concentration on the inner wall

Nusselt Number

$$Nu = \frac{hL}{k}$$

is the dimensionless form of the convection heat transfer coefficient and provides a measure of the convection heat transfer at a solid surface.

Mach Number

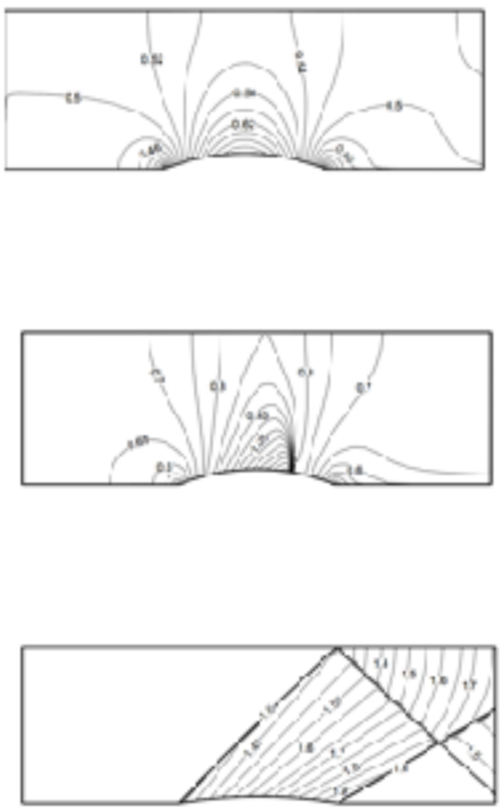
- Ma < 0.3 incompressible
- Otherwise compressible
- Mach number, Ma
  - Ma < 1 , subsonic.
  - Ma ≈ 1, transonic
  - M > 1 , supersonic (shock waves)
  - Ma > 5, hypersonic (high temperatures)
- These distinction affect the mathematical nature of the problem and therefore the solution method

$$Ma = \frac{|\mathbf{v}|}{a} = \frac{Flow\ speed}{Sound\ seed}$$

$$a = \sqrt{\gamma \left(\frac{\partial p}{\partial \rho} \right)_T}$$

For an ideal gas, it reduces to

$$a = \sqrt{\gamma RT}$$



(a) subsonic, (b) transonic, and (c) supersonic speeds.

Eckert Number

$$Ec = \frac{\mathbf{v} \cdot \mathbf{v}}{c_p \Delta T}$$

The Eckert number is a dimensionless number relating the kinetic energy of the flow to its enthalpy and is computed as

A large value of indicates high viscous dissipation occurring at high speed of the flow (high kinetic energy).

For small Eckert number several the terms in the energy equation become negligible (e.g., viscous dissipation, body forces, etc.). This reduces the energy equation to its incompressible form (i.e., a balance between conduction and convection).

$$\left(Ec \ll 1\right)$$

Froude Number

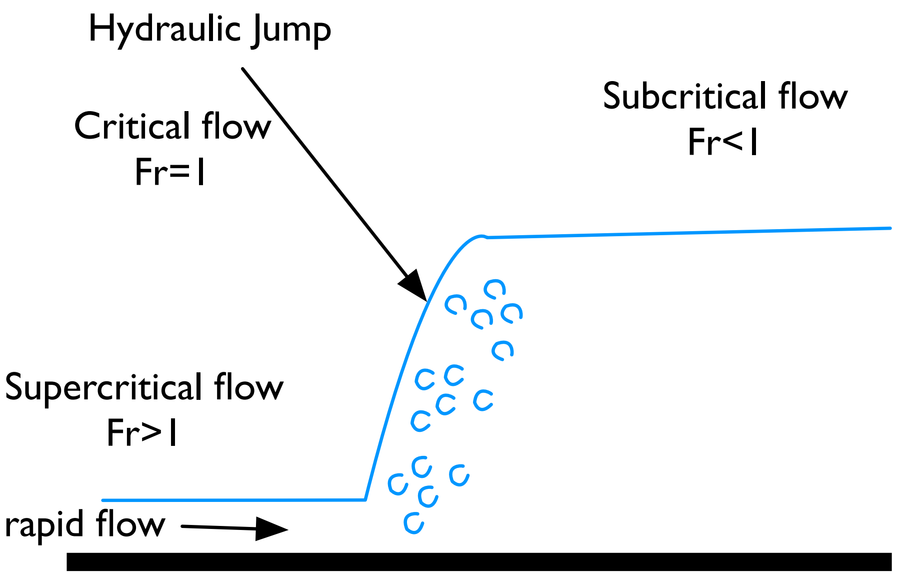
$$Fr = \frac{U}{c} = \frac{Velocity}{Wave\ \ propagation\ \ velocity}$$

$$Fr = \frac{U}{\sqrt{gL}}$$

$c = \sqrt{gL}$ $\quad L: length\ \ of\ \ ship\ \ at\ \ the\ \ waterline\ \ level$

$c = \sqrt{g\frac{A}{B}}$ $\quad A: cross-sectional\ \ area,\ B: free-surface\ \ width$

- Fr < 1 subcritical flow
- Fr > 1 supercritical flow
- Fr ≈ 1 critical flow



Weber Number

$$We = \frac{\rho U^2 L}{\sigma}$$

The Weber number represents the ratio of inertia to surface tension forces, is helpful in analyzing multiphase flows involving interfaces between two different fluids, with curved surfaces such as droplets and bubbles