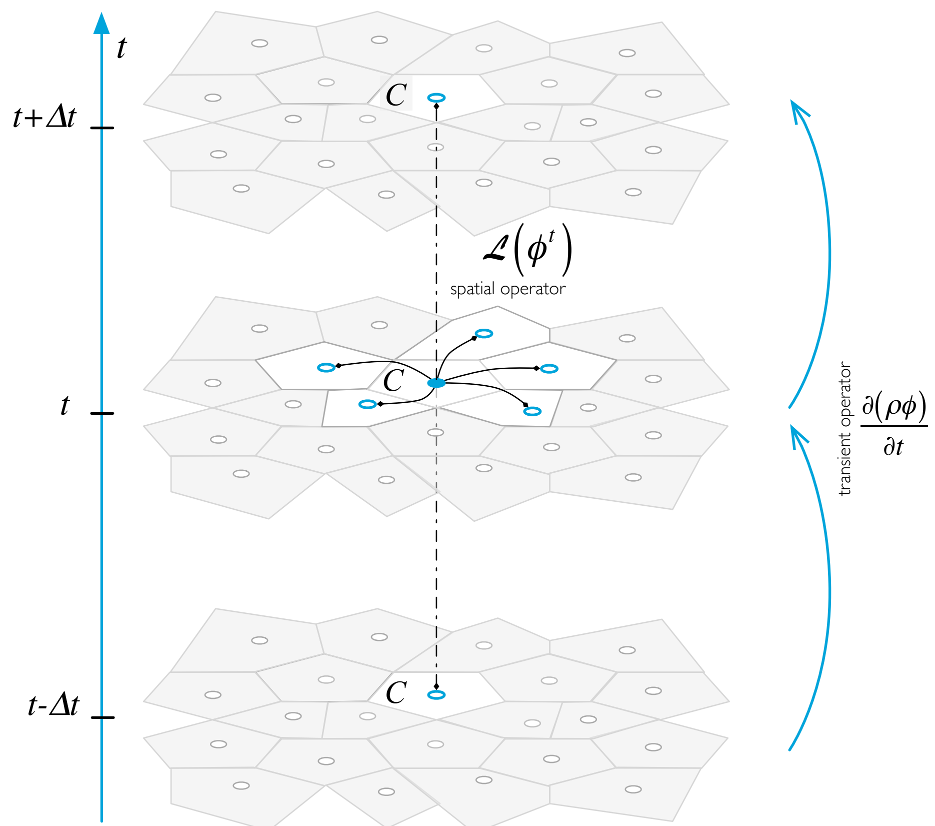
Transient, Source Terms and Relaxation

Chapter 13

Transient Term Discretization

Transient coordinate

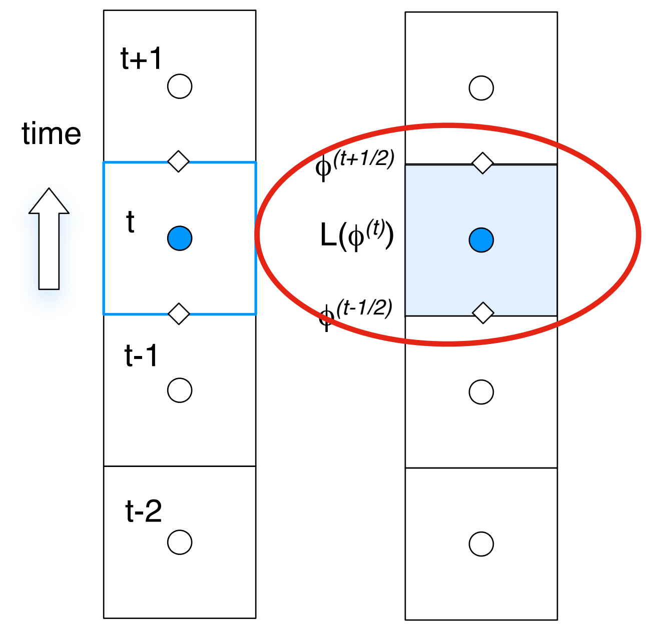


Transient Problems

transient term

$$\frac{\partial(\rho\phi)}{\partial t} = L(\phi)$$

spatial terms



integrate over spatial control volume

$$\int_\Omega \frac{\partial(\rho\phi)}{\partial t} dV = \int_\Omega L(\phi) dV \qquad \Rightarrow \frac{\partial(\rho\phi)}{\partial t} V = L(\phi) V$$

integrate over temporal control volume

$$\int_{t-\frac{1}{2}\Delta t}^{t+\frac{1}{2}\Delta t} \left(\frac{\partial(\rho\phi)}{\partial t} V \right) dt = \int_{t-\frac{1}{2}\Delta t}^{t+\frac{1}{2}\Delta t} \left(L(\phi) V \right) dt$$

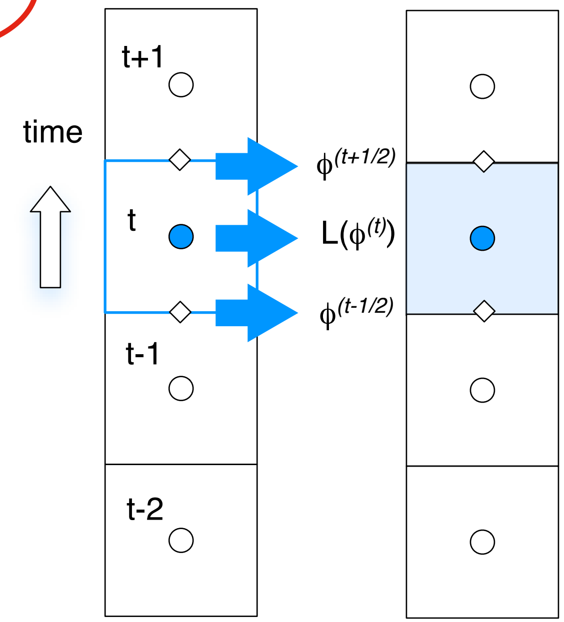
$$\left[(\rho\phi)^{t+\frac{1}{2}\Delta t} - (\rho\phi)^{t-\frac{1}{2}\Delta t} \right] V = \left[L(\phi) V \right] \Delta t \qquad \Rightarrow \frac{(\rho\phi)^{t+\frac{1}{2}\Delta t} - (\rho\phi)^{t-\frac{1}{2}\Delta t}}{\Delta t} V = L(\phi) V$$

Order of Discretization

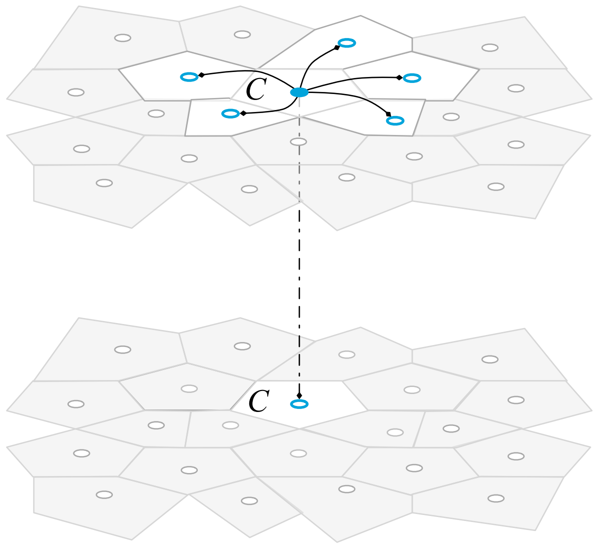
$$\frac{(\rho\phi)^{t+\frac{1}{2}\Delta t}-(\rho\phi)^{t-\frac{1}{2}\Delta t}}{\Delta t}V = L(\phi)V$$

$$\overline{L(\phi)V}^{t-\frac{1}{2}\Delta t\rightarrow t+\frac{1}{2}\Delta t} = L(\phi)V$$

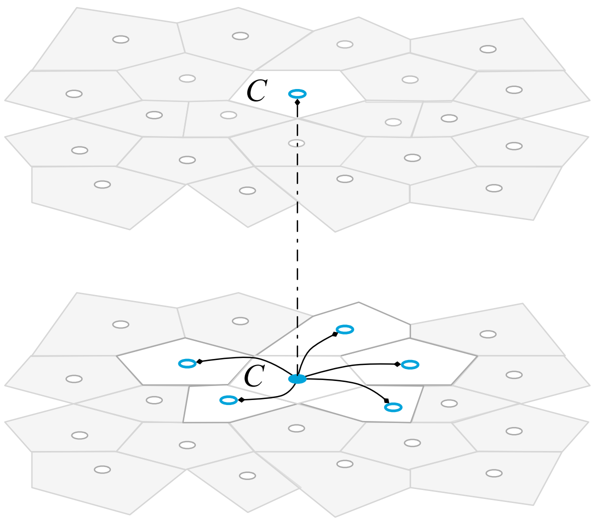
$$(\rho\phi)^{t+\frac{1}{2}\Delta t} = f\left(\rho\phi^{t+1},\rho\phi^{t},\rho\phi^{t-1},\rho\phi^{t-2},\ldots\right)$$



Implicit vs Explicit



backward Euler



Forward Euler

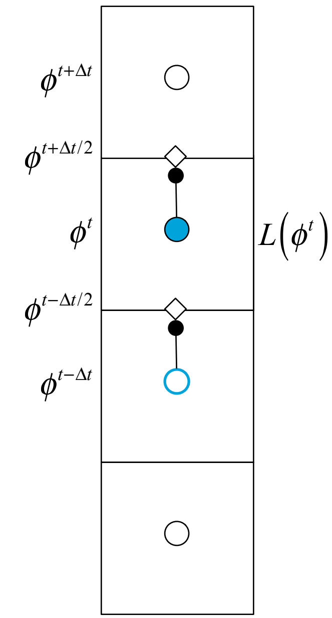
Backward Euler

$$\frac{(\rho\phi)^{t+\frac{1}{2}\Delta t} - (\rho\phi)^{t-\frac{1}{2}\Delta t}}{\Delta t} V = L(\phi) V$$

$$(\rho\phi)^{t+\frac{1}{2}\Delta t} \leftarrow (\rho\phi)^{t}$$

$$(\rho\phi)^{t-\frac{1}{2}\Delta t} \leftarrow (\rho\phi)^{t-\Delta t}$$

$$\Rightarrow \frac{(\rho\phi)^{t} - (\rho\phi)^{t-\Delta t}}{\Delta t} V = \left[L(\phi) V\right]^{t}$$



Implicit Scheme

Solve system of equations

Iterate

Stable for all Δt

Accuracy

$$(\rho\phi)^{t-\Delta t} = (\rho\phi)^{t} - \frac{\partial(\rho\phi)}{\partial t}\Delta t + \frac{\partial^2(\rho\phi)}{\partial t^2}\frac{\Delta t^2}{2} - \frac{\partial^3(\rho\phi)}{\partial t^4}\frac{\Delta t^3}{6} + \ldots.$$

$$\Rightarrow \frac{(\rho\phi)^{t} - (\rho\phi)^{t-\Delta t}}{\Delta t} = \frac{\partial(\rho\phi)}{\partial t} - \frac{\partial^2(\rho\phi)}{\partial t^2}\frac{\Delta t}{2} + \frac{\partial^3(\rho\phi)}{\partial t^4}\frac{\Delta t^2}{6} + \ldots.$$

first order accuracy

Numerical diffusion

$$\frac{\partial(\rho\phi)}{\partial t} = \left[L(\phi)V\right]^{t} + \underbrace{\frac{\Delta t}{2}\frac{\partial^2(\rho\phi)}{\partial t^2} + \ldots}_{\text{neglected terms}}$$

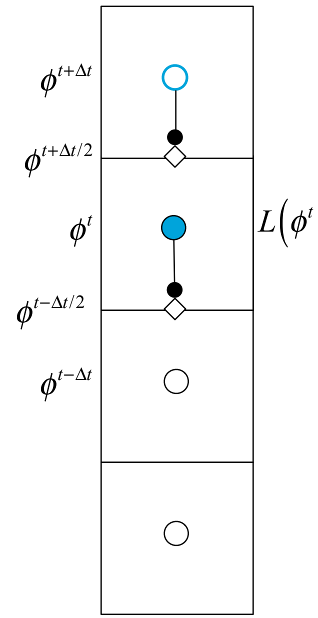
Forward Euler

$$\frac{(\rho\phi)^{t+\frac{1}{2}\Delta t} - (\rho\phi)^{t-\frac{1}{2}\Delta t}}{\Delta t} V = L(\phi)V$$

$$(\rho\phi)^{t+\frac{1}{2}\Delta t} \leftarrow (\rho\phi)^{t+\Delta t}$$

$$(\rho\phi)^{t-\frac{1}{2}\Delta t} \leftarrow (\rho\phi)^{t}$$

$$\Rightarrow \frac{(\rho\phi)^{t+\Delta t} - (\rho\phi)^{t}}{\Delta t} V = \left[L(\phi)V\right]^{t}$$



Explicit Scheme

Point Evaluation

No Iterations

Unstable for Courant>1

$$(\rho\phi)^{t+\Delta t} = (\rho\phi)^{t} + L(\phi)\Delta t$$

Accuracy

$$(\rho\phi)^{t+\Delta t} = (\rho\phi)^{t} + \frac{\partial(\rho\phi)}{\partial t}\Delta t + \frac{\partial^2(\rho\phi)}{\partial t^2}\frac{\Delta t^2}{2} + \frac{\partial^3(\rho\phi)}{\partial t^4}\frac{\Delta t^3}{6} + \ldots.$$

$$\Rightarrow \frac{(\rho\phi)^{t+\Delta t} - (\rho\phi)^{t}}{\Delta t} = \frac{\partial(\rho\phi)}{\partial t} + \frac{\partial^2(\rho\phi)}{\partial t^2}\frac{\Delta t}{2} + \frac{\partial^3(\rho\phi)}{\partial t^4}\frac{\Delta t^2}{6} + \ldots.$$

first order accuracy

$$\frac{\partial(\rho\phi)}{\partial t} = \left[L(\phi)V\right]^{t} \underbrace{- \frac{\Delta t}{2}\frac{\partial^2(\rho\phi)}{\partial t^2} + \ldots}_{\text{neglected terms}}$$

Numerical anti-diffusion

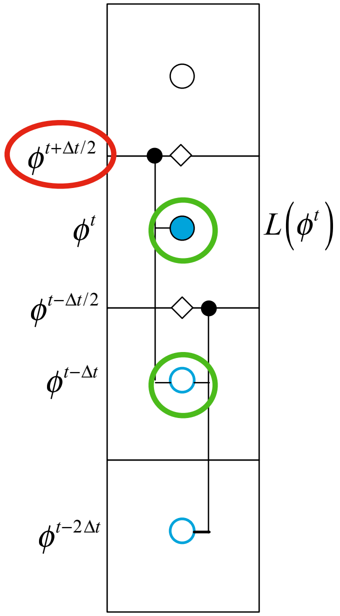
Adam-Bashforth

$$\frac{(\rho\phi)^{t+\frac{1}{2}\Delta t} - (\rho\phi)^{t-\frac{1}{2}\Delta t}}{\Delta t} V = L(\phi) V$$

$$(\rho\phi)^{t+\frac{1}{2}\Delta t} \leftarrow \frac{3}{2}(\rho\phi)^{t} - \frac{1}{2}(\rho\phi)^{t-\Delta t}$$

$$(\rho\phi)^{t-\frac{1}{2}\Delta t} \leftarrow \frac{3}{2}(\rho\phi)^{t-\Delta t} - \frac{1}{2}(\rho\phi)^{t-2\Delta t}$$

$$\Rightarrow \frac{3(\rho\phi)^{t} - 4(\rho\phi)^{t-\Delta t} + 2(\rho\phi)^{t-2\Delta t}}{2\Delta t} V = \left[L(\phi) V\right]^{t}$$



Implicit Scheme

2 old time-steps needed

Second Order Accuracy

Crank-Nicholson

$$\frac{(\rho\phi)^{t+\frac{1}{2}\Delta t} - (\rho\phi)^{t-\frac{1}{2}\Delta t}}{\Delta t} V = L(\phi) V$$

$$(\rho\phi)^{t+\frac{1}{2}\Delta t} \leftarrow \frac{1}{2}(\rho\phi)^{t+\Delta t} + \frac{1}{2}(\rho\phi)^{t}$$

$$(\rho\phi)^{t-\frac{1}{2}\Delta t} \leftarrow \frac{1}{2}(\rho\phi)^{t} + \frac{1}{2}(\rho\phi)^{t-\Delta t}$$

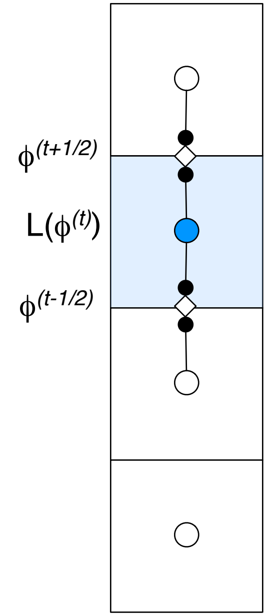
$$\Rightarrow \frac{(\rho\phi)^{t+\Delta t} + 2(\rho\phi)^{t-\Delta t}}{2\Delta t} V = \left[L(\phi) V \right]^{t}$$

Explicit Scheme

2 old time-steps needed

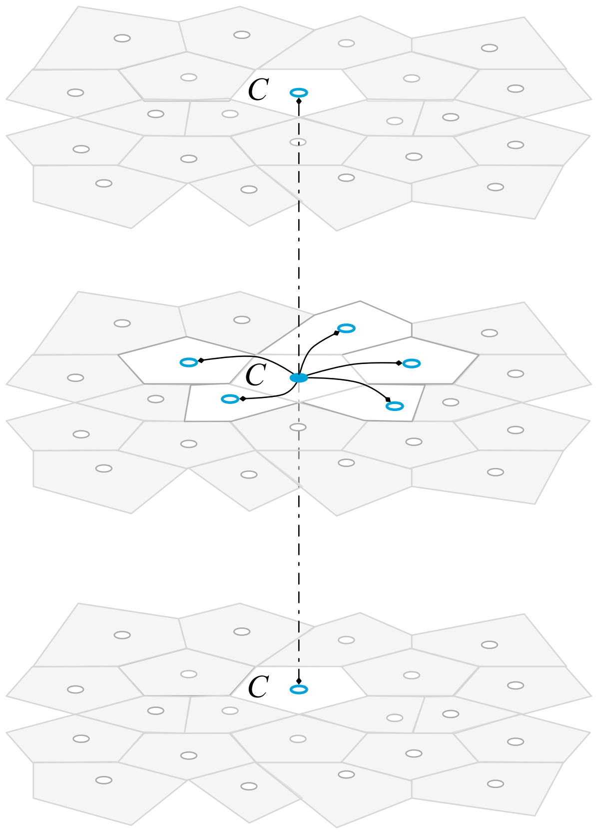
Second Order Accuracy

Unstable for Courant>2



Crank-Nicholson

Crank-Nicholson

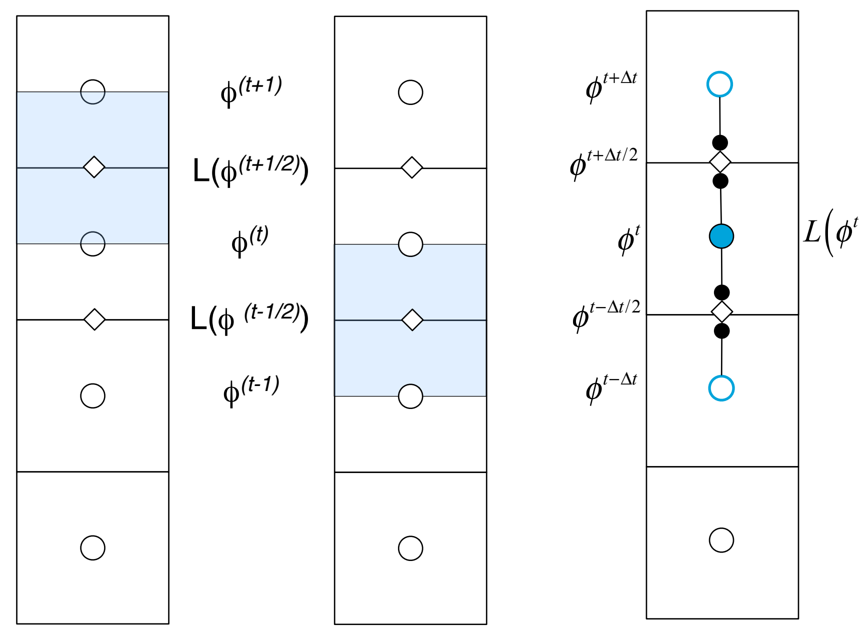


Crank-Nicholson-Implementation

Backward Euler $\Rightarrow \dfrac{(\rho\phi)^{t}-(\rho\phi)^{t-\Delta t}}{\Delta t}V=\left[L(\phi)V\right]^{t}$

Forward Euler $\Rightarrow \dfrac{(\rho\phi)^{t+\Delta t}-(\rho\phi)^{t}}{\Delta t}V=\left[L(\phi)V\right]^{t}$

Crank-Nicholson $\dfrac{(\rho\phi)^{t+\Delta t}-(\rho\phi)^{t-\Delta t}}{\Delta t}V=2\left[L(\phi)V\right]^{t}$



(1) $\dfrac{(\rho\phi)^{t}-(\rho\phi)^{t-\Delta t}}{\Delta t}V=\left[L(\phi)V\right]^{t}$

Note that $\dfrac{(\rho\phi)^{t+\Delta t}-(\rho\phi)^{t}}{\Delta t}V=\left[L(\phi)V\right]^{t}=\dfrac{(\rho\phi)^{t}-(\rho\phi)^{t-\Delta t}}{\Delta t}V$

Thus step 2 can be written as

(2) $(\rho\phi)^{t+\Delta t}=2(\rho\phi)^{t}-(\rho\phi)^{t-\Delta t}$

Stability of the Explicit Scheme

$$\frac{\partial(\rho\phi)}{\partial t} = \nabla \cdot \Gamma \nabla \phi + Q \qquad \Rightarrow \frac{(\rho\phi)^{t} - (\rho\phi)^{t-\Delta t}}{\Delta t} V = \sum_{nb} (\Gamma \nabla \phi)_f \cdot S_f + QV$$

$$a_P^t \phi_P^t = \sum_{NB} a_N^{t-\Delta t} \phi_N^{t-\Delta t} + \left(a_P^{t-\Delta t} - \sum_{NB} a_N^{t-\Delta t} \right) \phi_P^{t-\Delta t} + b_P$$

$$if \left(a_P^{t-\Delta t} - \sum_{NB} a_{NB}^{t-\Delta t} \right) < 0 \Rightarrow Unphysical \;\; feedback$$

$$a_P^t = \frac{\rho^t}{\Delta t} V \qquad a_P^{t-\Delta t} = \frac{\rho^{t-\Delta t}}{\Delta t} V$$

$$b_P = QV$$

$$a_E^{t-\Delta t} = \frac{\Gamma_e^{t-\Delta t} \Delta y}{\delta x} \qquad a_W^{t-\Delta t} = \frac{\Gamma_w^{t-\Delta t} \Delta y}{\delta x}$$

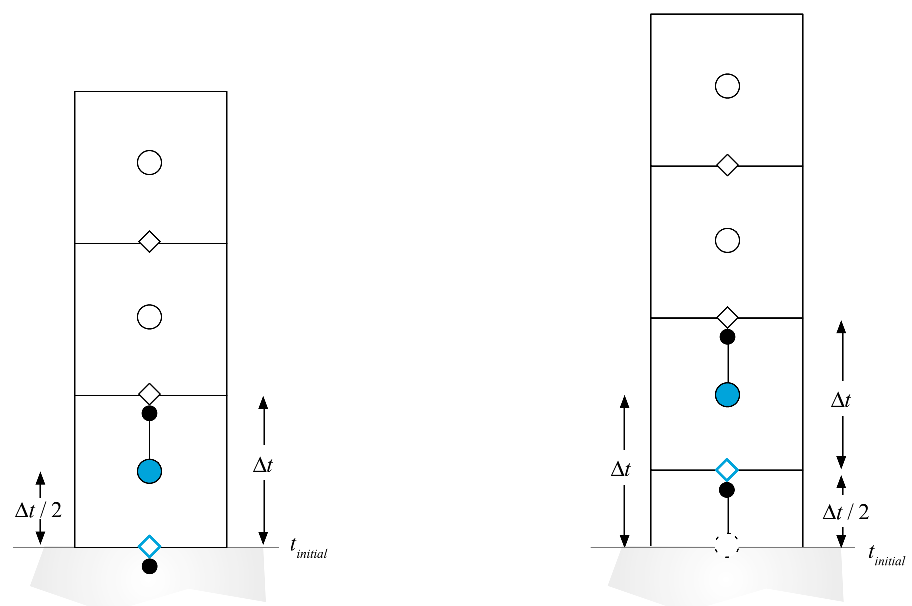
$$a_N^{t-\Delta t} = \frac{\Gamma_n^{t-\Delta t} \Delta x}{\delta y} \qquad a_S^{t-\Delta t} = \frac{\Gamma_s^{t-\Delta t} \Delta x}{\delta y}$$

$$1D: \frac{\rho \Delta x}{\Delta t} \geq \left(\frac{\Gamma_e \Delta y}{\delta x_e} + \frac{\Gamma_w \Delta y}{\delta x_w} \right) \Rightarrow \Delta t \leq \frac{\rho (\Delta x)^2}{2\Gamma}$$

$$2D: \frac{\rho \Delta x \Delta y}{\Delta t} \geq \left(\frac{\Gamma_e \Delta y}{\delta x_e} + \frac{\Gamma_w \Delta y}{\delta x_w} + \frac{\Gamma_n \Delta x}{\delta y_n} + \frac{\Gamma_s \Delta x}{\delta y_s} \right) \Rightarrow \Delta t \leq \frac{\rho (\Delta x)^2}{4\Gamma}$$

Von Neumann Stability Criterion

Initial Condition



Conclusion