Solving the Navier-Stokes Equations

Chapter 16

Pressure Equation for Compressible Flow

Compressible Flow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \nabla \cdot \tau - \nabla p + \mathbf{B}$$

$$\rho = C_\rho p$$

$$\rho\big|_{(P^* + P')} = \rho\big|_{(P)} + \frac{\partial \rho}{\partial P} p' \Rightarrow \rho' = \left(\frac{\partial \rho}{\partial P}\right) p' = C_\rho p'$$

$$p = p^{(n)} + p'$$

$$\rho = \rho^{(n)} + \rho'$$

$$\mathbf{v} = \mathbf{v}^* + \mathbf{v}'$$

Discretized Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Incompressible

$$\sum_{f=nb(P)} \left(\dot{m}_f^* + \dot{m}_f' \right) = 0$$

$$\dot{m}_f = \rho_f \left(\mathbf{v}_f^\bullet + \mathbf{v}_f' \right) \cdot \mathbf{S}_f = \underbrace{\rho_f \mathbf{v}_f^\bullet \cdot \mathbf{S}_f}_{\dot{m}_f^\bullet} + \underbrace{\rho_f \mathbf{v}_f' \cdot \mathbf{S}_f}_{\dot{m}_f'}$$

Compressible

$$\frac{\left(\rho_P + \rho_P' - \rho_P^{(n)} \right)}{\Delta t} V_P + \sum_{f=nb(P)} \left(\dot{m}_f^* + \dot{m}_f' \right) = 0$$

$$\dot{m}_f = \left(\rho_f^{(n)} + \rho_f' \right) \left(\mathbf{v}_f^\bullet + \mathbf{v}_f' \right) \cdot \mathbf{S}_f$$

$$= \underbrace{\rho_f^{(n)} \mathbf{v}_f^\bullet \cdot \mathbf{S}_f}_{\dot{m}_f^\bullet} + \underbrace{\rho_f^{(n)} \mathbf{v}_f' \cdot \mathbf{S}_f + \rho_f' \mathbf{v}_f^\bullet \cdot \mathbf{S}_f + \rho_f' \mathbf{v}_f' \cdot \mathbf{S}_f}_{\dot{m}_f'}$$

Velocity Correction

$$\underbrace{\rho_f^{(n)}\mathbf{v}_f^{\bullet}\cdot\mathbf{S}_f}_{\dot{m}_f^{\bullet}} + \underbrace{\rho_f^{(n)}\mathbf{v}'_f\cdot\mathbf{S}_f + \rho'_f\mathbf{v}_f^{\bullet}\cdot\mathbf{S}_f + \rho'_f\mathbf{v}'_f\cdot\mathbf{S}_f}_{\dot{m}'_f}$$

⇨ $$\dot{m}_f^{\bullet} = \rho_f^{(n)}\overline{\mathbf{v}_f^{\bullet}}\cdot\mathbf{S}_f - \rho_f^{(n)}\overline{\mathbf{D}_f}\left(\nabla p_f^{(n)} - \overline{\nabla p_f^{(n)}}\right)\cdot\mathbf{S}_f$$

⇨ $$\dot{m}'_f = \underbrace{\rho_f^{(n)}\overline{\mathbf{v}'_f}\cdot\mathbf{S}_f - \rho_f^{(n)}\overline{\mathbf{D}_f}\left(\nabla p'_f - \overline{\nabla p'_f}\right)\cdot\mathbf{S}_f}_{(1)} + \underbrace{\left(\frac{\dot{m}_f^{\bullet}}{\rho_f^{(n)}}\cdot\mathbf{S}_f\right)C_{\rho,f}p'_f}_{(2)}$$

Pressure Equation

$$\frac{\left(\rho_C + \rho'_C - \rho_C^{(n)}\right)}{\Delta t} V_C + \sum_{f=nb(C)} \left(\dot{m}_f^{\bullet} + \dot{m}'_f\right) = 0$$

$$\frac{V_P}{\Delta t} C_\rho p'_C + \sum_{f=nb(P)} \left\{ -\rho_f^{(n)} \overline{\mathbf{D}}_f \nabla p'_f \cdot \mathbf{S}_f + \left(\frac{\dot{m}_f^{\bullet}}{\rho_f^{(n)}} \right) C_\rho p'_f \right\} =$$

$$-\left(\frac{\rho_C^{(n)} - \rho_C^{o}}{\Delta t} V_C + \sum_{f=nb(C)} \dot{m}_f^{\bullet} \right) - \underline{\sum_{f=nb(C)} \rho_f^{(n)} \left(\overline{\mathbf{v}'_f} + \overline{\mathbf{D}}_f \overline{\nabla p'_f} \right) \cdot \mathbf{S}_f} - \underline{\underline{\sum_{f=nb(C)} \rho'_f \mathbf{v}'_f \cdot \mathbf{S}_f}}$$

$$\sum_{f=nb(C)} \left(\rho'_f \overline{\mathbf{v}'_f} + \overline{\mathbf{D}}_f \overline{\nabla p'_f} \right) \cdot \mathbf{S}_f = \sum_{f=nb(P)} -\overline{\mathbf{H}}_f [\mathbf{v}'] \cdot \mathbf{S}_f = \sum_{f=nb(C)} -0.5 \left(\mathbf{H}'_C + \mathbf{H}'_N \right) \cdot \mathbf{S}_f$$

$$= \sum_{f=nb(C)} -0.5 \left(\sum_{NBP(C)} \left(\frac{a_{NBP}}{a_P} \mathbf{v}'_{NBP} \right) + \sum_{NB(F)} \left(\frac{a_{NBF}}{a_F} \mathbf{v}'_{NBF} \right) \right) \cdot \mathbf{S}_f$$

Pressure Equation

$$\underbrace{\frac{V_C C_\rho}{\Delta t} p'_C}_{\text{transient-like term}} + \underbrace{\sum_{f=nb(C)} \left[C_\rho \left(\frac{\dot{m}_f^*}{\rho_f^{(n)}} \right) p'_f \right]}_{\text{convection-like term}} + \underbrace{\sum_{f=nb(C)} \left[-\rho_f^{(n)} \mathbf{D}_f (\nabla p')_f \cdot \mathbf{S}_f \right]}_{\text{diffusion-like term}} = \underbrace{-\left(\frac{\left(\rho_C^{(n)} - \rho_C^\circ\right)}{\Delta t} V_C + \sum_{f=nb(C)} \dot{m}_f^* \right)}_{\text{source-like term}}$$

$$\Rightarrow \sum_{f=nb(P)} \left[-\rho_f^{(n)} \mathbf{D}_f (\nabla p')_f \cdot \mathbf{S}_f \right] = \sum_{f=nb(P)} \left[-\rho_f^{(n)} \mathbf{D}_f (\nabla p')_f \cdot \left(\mathbf{E}_f + \mathbf{T}_f \right) \right] = \sum_{f=nb(P)} \left[-\rho_f^{(n)} \mathbf{D}_f \cdot \left(p'_F - p'_C \right) \right]$$

$$\Rightarrow \sum_{f=nb(C)} \left[C_\rho \left(\frac{\dot{m}_f^*}{\rho_f^{(n)}} \right) p'_f \right] = \left\| \dot{m}_f^\bullet, 0 \right\| \frac{C_{\rho,f}}{\rho_f^{(n)}} p'_C - \left\| -\dot{m}_f^\bullet, 0 \right\| \frac{C_{\rho,f}}{\rho_f^{(n)}} p'_F$$

$$\boldsymbol{D}_f = \left(\frac{\overline{d_f^u} E_{x,f} + \overline{d_f^v} E_{y,f}}{\left\| \mathbf{d}_{PF} \right\|} \right)$$

$$a_C = \frac{V_C C_\rho}{\Delta t} + \sum_{f=nb(C)} \left(\frac{C_{\rho,f}}{\rho_f^{(n)}} \left\| \dot{m}_f^*, 0 \right\| \right) + \sum_{f=nb(C)} \rho_f^{(n)} \mathbf{D}_f$$

$$a_F = -\left\| -\dot{m}_f^*, 0 \right\| \frac{C_{\rho,f}}{\rho_f^{(n)}} - \rho_f^{(n)} \mathbf{D}_f$$

$$b_C = -\left(\frac{\left(\rho_C^{(n)} - \rho_C^\circ\right)}{\Delta t} V_C + \sum_{f=nb(C)} \dot{m}_f^* \right) + \sum_{f=nb(C)} \rho_f^{(n)} \left(\mathbf{D}_f \nabla p'_f \right) \cdot \mathbf{T}_f$$

Pressure Equation

$$\frac{V_C C_\rho}{\Delta t}\left(p'_C\right) + \sum_f \left[C_\rho U^\bullet_f p'_f + \rho^\bullet_f \left(\underbrace{\overline{\mathbf{v}'_f} - \overline{\mathrm{D}}_f \left(\nabla p' - \overline{\nabla p'}\right)_f}_{\text{Rhie-Chow interpolation}} \right) \cdot \mathbf{S}_f \right] = -\frac{V_C}{\Delta t}\left(\rho^\bullet_C - \rho^\circ_C\right) - \sum_{f=nb(C)} \dot{m}^\bullet_f - \sum_{f=nb(C)} \left(\rho'_f \mathbf{v}'_f \cdot \mathbf{S}_f\right)$$

$$\frac{V_C C_\rho}{\Delta t}\left(p'_C\right) + \sum_{f=nb(C)} \left(C_\rho U^\bullet_f P'\right)_f - \sum_{f=nb(C)} \left(\rho^\bullet_f \overline{\mathrm{D}}_f \left(\nabla p'\right)_f \cdot \mathbf{S}_f\right) = -\frac{V_C}{\Delta t}\left(\rho^\bullet_C - \rho^\circ_C\right) - \sum_{f=nb(C)} \left(\rho^*_f U^*_f\right) - \underline{\sum_{f=nb(C)} \left(\rho'_f \mathbf{v}'_f \cdot \mathbf{S}_f\right)} - \underline{\underline{\sum_{f=nb(C)} \left[\rho^\bullet_f \left(\overline{\mathbf{v}'_f} + \overline{\mathrm{D}}_f \left(\overline{\nabla P'}\right)_f\right) \cdot \mathbf{S}_f\right]}}$$

High Resolution

Neglect

treatment leads to variety of schemes

SIMPLE
SIMPLEC
SIMPLER
SIMPLEST
SIMPLE-M
PISO

$$\frac{V_C C_\rho}{\Delta t}\left(p'_C\right) + \sum_{f=nb(C)} \left(C_\rho \frac{\dot{m}_f}{\rho^\bullet_f} p'_f\right) - \sum_{f=nb(C)} \left(\rho^\bullet_f \overline{\mathrm{D}}_f \left(\nabla p'\right)_f \cdot \mathbf{S}_f\right) = -\underbrace{\left(\frac{\Omega}{\Delta t}\left(\rho^\bullet_P - \rho^o_P\right) + \sum_{f=nb(C)} \dot{m}_f\right)}_{\text{Residual}}$$

$$a_C p'_C + \sum_{F=NB(C)} a_F p'_F = b_C \longrightarrow p^\bullet, \mathbf{v}^{\bullet\bullet}, \dot{m}^{\bullet\bullet}$$

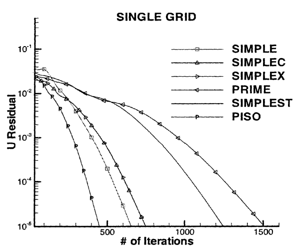
All-Speed Flow Algorithms

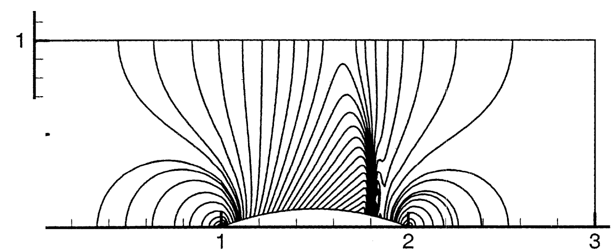
Transient-like Term

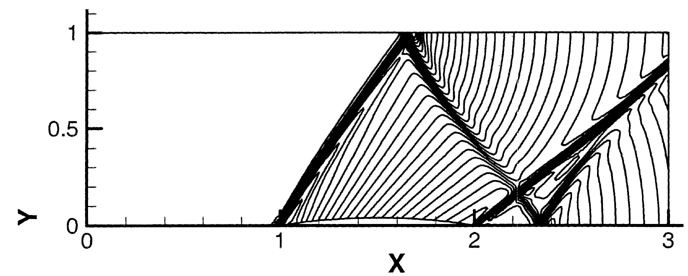
$$\frac{\Omega C_\rho}{\Delta t}(P'_P) + \sum_f \left(C_\rho U^*_f P'\right) - \sum_f \left(\rho^*_f \overline{\mathrm{D}}_f (\nabla P')_f \cdot \mathbf{S}_f\right) = -\left[\frac{\Omega}{\Delta t}(\rho^*_P - \rho^o_P) + \sum_f \left(\rho^*_f U^*_f\right)\right] - \sum_f \left(\rho'_f \mathbf{v}'_f \cdot \mathbf{S}_f\right) - \sum_f \left(\rho^*_f \overline{\mathrm{H}[\mathbf{v}']}_f \cdot \mathbf{S}_f\right)$$

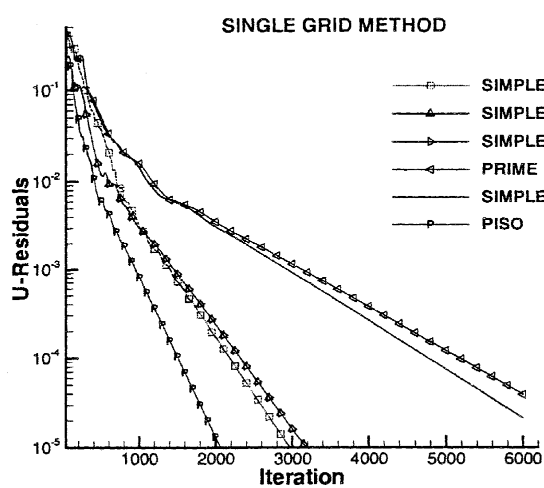
Advection-like Term

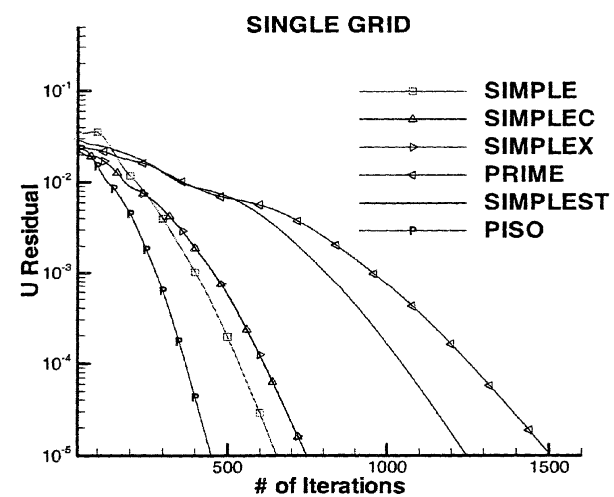
account for compressibility effects



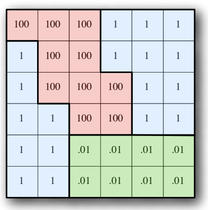


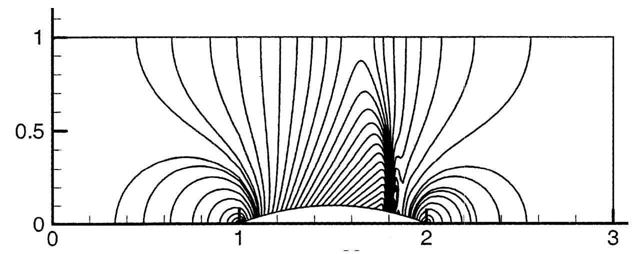


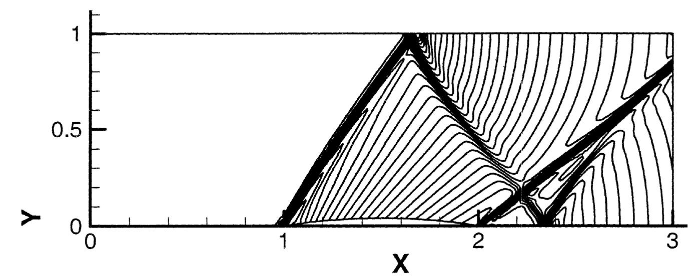


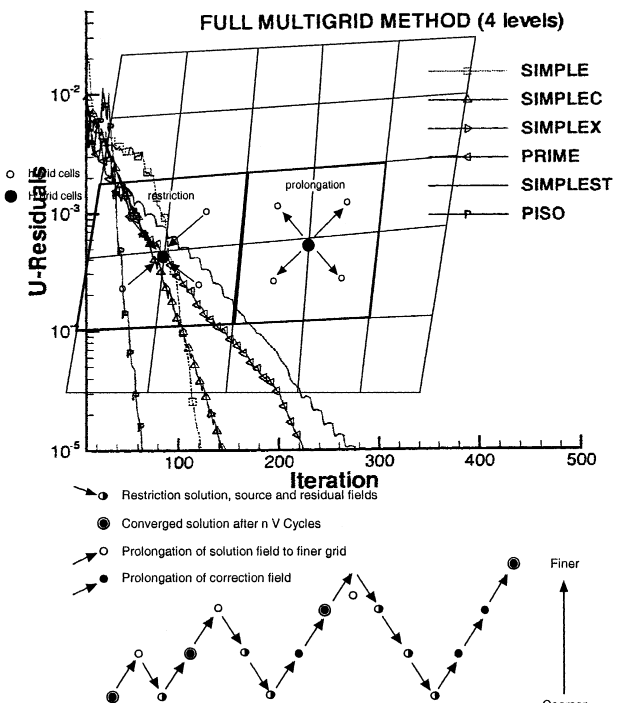


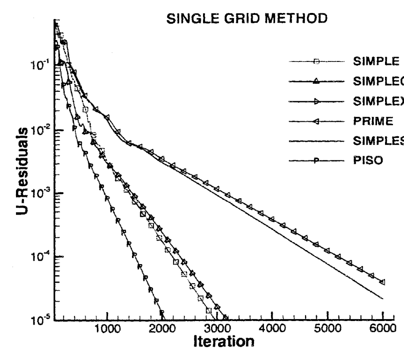
Multigrid Acceleration

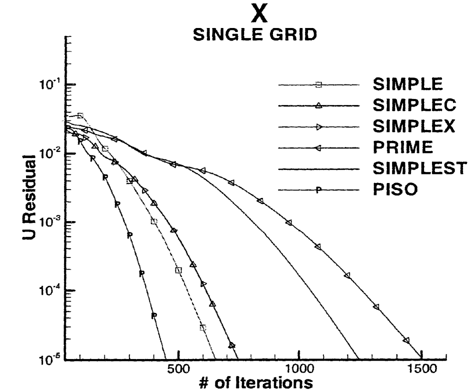


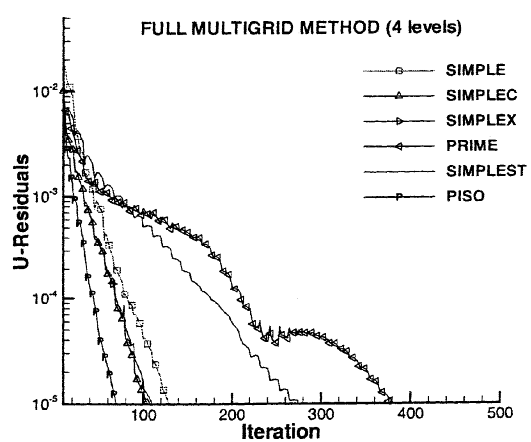


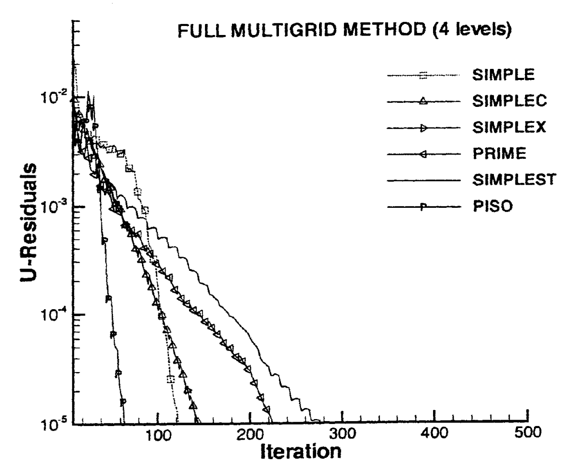












Problem 1- Staggered Grid

- Use the SIMPLE procedure to compute p2, uB, and uC from the following data:

$$\Delta x = 2,\ c_B = 0.25, c_C = 0.2$$

$$A_B = 5, A_C = 4, p_1 = 200, p_3 = 38$$

- As an initial guess, set

$$u_B = u_C = 15 \text{ and } p_2 = 120$$