

THE EFFICIENCY OF ENDOREVERSIBLE HEAT ENGINES WITH HEAT LEAK

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SUMMARY

Finite-time thermodynamics are used for studying the performance of endoreversible heat engines with heat leak. A comprehensive formulation and a general solution methodology, valid for any mode of heat supply or release, are presented. Detailed analyses are conducted for several heat transfer modes and universal analytical and numerical results for the efficiency at maximum power are generated. Many established laws and major conclusions derived in several references are shown to represent very special cases of the new formulation. Furthermore, the nature of the leakage power law is found to deeply affect the efficiency at maximum power. Finally, for no leakage situations, if the heat to the engine is supplied and released via similar heat transfer modes, then the lowest efficiency at maximum power, when the only thermal resistance is between the working fluid and the hot reservoir, is found to be given by $1/n$, n being the power of the heat transfer law.

KEY WORDS endoreversible thermodynamics; Carnot engines; heat leak; maximum energy conversion

INTRODUCTION

The efficiency of endoreversible heat engines at maximum power conditions has been the subject of several investigations. Many variations in modelling the problem and in approaching the solution have been considered (Curzon and Ahlborn, 1975; Chen and Yan, 1989; DeVos, 1985). The common feature of the various reported studies is that they have dealt with the additional limitation on efficiency caused by the rate at which heat can be exchanged between the working material and the heat reservoirs. This additional resistance to heat flow, was first accounted for by Curzon and Ahlborn (1975) who considered the heat flux across the walls of the hot and cold reservoirs to be proportional to the prevailing temperature difference there (i.e. Newton's law). Chen and Yan (1989) generalized the work reported in Curzon and Ahlborn (1975) by assuming the rate of heat flowing through the walls of the reservoirs to be ruled by an equation of the form:

$$Q = \alpha(T_1^n - T_2^n) \quad (1)$$

where n is a nonzero integer. The use of equal powers to describe the rates of heat in and out has limited the applicability of results to those situations where similar heat transfer modes govern the hot and cold sides of the engine. DeVos (1985) simplified the analysis presented in Curzon and Ahlborn (1975) and Chen and Yan (1989) by developing an easier model. Even though this model was introduced in general terms, only specific cases of limited usefulness were analysed.

To further elaborate, endoreversible thermodynamics have also been applied by several workers (Gordon, 1991; Gordon and Zarmi, 1989; DeVos, 1991; Nuwayhid and Moukalled, 1994) in studying and predicting phenomena of practical interest. Gordon (1991) applied finite-time thermodynamics to analyse the thermoelectric generator. Gordon and Zarmi (1989) modelled the earth and its envelope using a Carnot engine with its heat input being solar radiation and its work output representing the wind generated. From these basic considerations, they derived a theoretical upper bound for the annual average wind energy on earth. DeVos (1991) developed a simplified version of Gordon and Zarmi's model

and applied it to studying the conversion efficiency of solar energy into wind energy. Nuwayhid and Moukalled (1994) added a heat leak term into the model of DeVos (1991) and studied the effect of a planet thermal conductance on conversion efficiency of solar energy into wind energy. The theoretical upper bound on conversion efficiency reported in DeVos (1991) was shown to be well above the actual values predicted by the modified model. Recently, Nulton *et al.* (1993) and Pathria *et al.* (1993) described a set of feasible operations of a finite-time heat engine subject only to thermal losses in terms of an inequality similar to the second law of thermodynamics and applied it to Carnot-like refrigerators and heat pumps.

From the above literature survey it appears that, even though heat leak has a realistic influence on model performance (Nuwayhid and Moukalled, 1994) it has not been widely exploited. Adding a heat-leak term into the model, makes it more realistic since, it serves as a mean to account for the losses occurring within the engine and between the engine and the surroundings in a reversible manner. Furthermore, a general solution of the Curzon–Ahlborn concept has not been attempted yet. It is the intention of this paper to generalize the Curzon–Ahlborn concept and to make it more realistic by adding a heat leak term into the DeVos model (1991). As shown in Figure 1, the heat-leak term is itself dependent upon the heat transfer mode. As will be seen later, this generalization results in equations which, in some cases, turn out to be quite complex and have to be tackled numerically. Moreover, a variety of well established formulae, such as the Curzon–Ahlborn efficiency, the Castans efficiency, etc., are shown to represent very special cases of the general results presented in this work.

THE HEAT-LEAK MODEL

A schematic of the endoreversible engine under consideration is depicted in Figure 1. Irreversible heat transfer takes place between the heat source and the hot reservoir of the engine, while heat transfer from

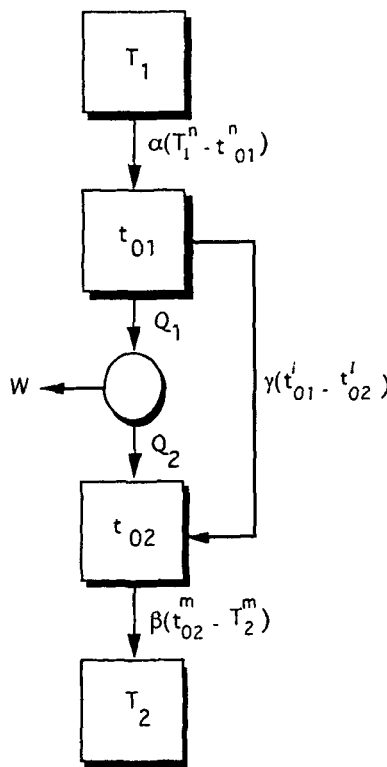


Figure 1. Schematical model of endoreversible engine with heat leak operating between hot reservoir at T_1 and cold reservoir at T_2

the hot reservoir to the engine is considered to occur with no resistance (reversible). An analogous situation exists between the engine and the heat sink. The friction loss within the engine and the heat loss between the engine and the surroundings are reversibly modelled via the heat-leak term. With such a model, the heat transfer from the hot reservoir into the engine Q_1 is given by a heat balance equation as

$$Q_1 = \alpha(T_1^n - t_{01}^n) - \gamma(t_{01}^l - t_{02}^l) \quad (2)$$

while the heat transfer from the engine to the cold reservoir is

$$Q_2 = \beta(t_{02}^m - T_2^m) - \gamma(t_{01}^l - t_{02}^l) \quad (3)$$

where α , β , and γ are the appropriate coefficients of heat transfer and n , m , and l are the model powers of the heat transfer laws (e.g., they are 1 for conduction/convection and 4 for radiation).

Endoreversibility requires that

$$\frac{Q_1}{t_{01}} = \frac{Q_2}{t_{02}} \quad (4)$$

with the Carnot efficiency given by

$$\eta = 1 - \frac{t_{02}}{t_{01}} \quad (5)$$

The work can therefore be found from the following relation:

$$W = \alpha(T_1^n - t_{01}^n) - \beta(t_{02}^m - T_2^m) \quad (6)$$

Applying the reversibility condition by inserting Q_1 and Q_2 into equation (4), the following equation for t_{01} in terms of T_1 , T_2 , α , β and γ obtained:

$$\alpha(1 - \eta)t_{01}^n + \beta(1 - \eta)^m t_{01}^m - \gamma[\eta + (1 - \eta)^{l+1} - (1 - \eta)^l]t_{01}^l - \alpha(1 - \eta)T_1^n - \beta T_2^m = 0 \quad (7)$$

Defining the following dimensionless quantities:

$$R = \frac{\beta T_2^m}{\alpha T_1^n}, \quad S = \frac{\gamma T_1^l}{\alpha T_1^n}, \quad \tau = \frac{T_2}{T_1}, \quad \text{and} \quad t_1 = \frac{t_{01}}{T_1} \quad (8)$$

and using the Carnot efficiency (equation (5)), the efficiency and work equations (equations (7) and (6)) are transformed respectively to

$$(1 - \eta)t_1^n + R(1 - \eta)^m t_1^m - S[\eta + (1 - \eta)^{l+1} - (1 - \eta)^l]t_1^l - (1 - \eta) - R\tau^m = 0 \quad (9)$$

and

$$\frac{W}{\alpha T_1^n} = 1 + R\tau^m - t_1^n - R t_1^m (1 - \eta)^m \quad (10)$$

Analytical solutions to the above dimensionless equations may be obtained for some specific situations. In general however, the solutions should be obtained numerically.

SOLUTION METHODOLOGY

The general equations (9) and (10) are used in this section to obtain the efficiency of the endoreversible heat engine at maximum power. For this purpose, the derivative of the normalized power equation (equation (10)) with respect to the efficiency is set to zero. This results in the following general relation:

$$\frac{dt_1}{d\eta} = \frac{m(1 - \eta)^{m-1} t_1}{\left[\frac{n}{R} t_1^{n-m} + m(1 - \eta)^m \right]} \quad (11)$$

A second equation for the derivative of t_1 with respect to η may be obtained from the reversibility equation (9), is given by:

$$\frac{dt_1}{d\eta} = \frac{t_1^n + mR(1-\eta)^{m-1}t_1^m + S[1 - (l+1)(1-\eta)^l + l(1-\eta)^{l-1}]t_1^l - 1}{n(1-\eta)t_1^{n-1} + mR(1-\eta)^m t_1^{m-1} - S[\eta + (1-\eta)^{l+1} - (1-\eta)^l]l t_1^{l-1}} \quad (12)$$

Equating equations (11) and (12), a relation for the efficiency at maximum power (η_m) is obtained and its final form is written as

$$\begin{aligned} m(1-\eta_m)^{m-1} \left\{ n(1-\eta_m)t_1^n + mR(1-\eta_m)^m t_1^m - S[\eta_m + (1-\eta_m)^{l+1} - (1-\eta_m)^l]l t_1^l \right\} \\ = \left[\frac{n}{R}t_1^{n-m} + m(1-\eta_m)^m \right] \left\{ t_1^n + mR(1-\eta_m)^{m-1} t_1^m + S[1 - (l+1)(1-\eta_m)^l + l(1-\eta_m)^{l-1}]t_1^l - 1 \right\} \end{aligned} \quad (13)$$

At the same time, the efficiency should satisfy the reversibility equation (9). This results in a highly nonlinear system of two equations in the two unknowns t_1 and η_m . In general, the solution to the above system should be obtained numerically. However, for some special cases of practical interest, the above system may be reduced to a single equation in η_m for which the solution may be carried out numerically. Furthermore, earlier results reported in (Curzon and Ahlborn, 1975; Chen and Yen, 1989; DeVos, 1985) are shown to represent very special cases of this general formulation. These special cases are the subject of the following sections.

The general case $n = m$

For the case when $n = m$, the above nonlinear system of equations may be reduced analytically to a single equation in η_m which may always be solved numerically. However, depending on the values of R , S , and τ , analytical solutions may sometimes be possible. Under these conditions, equations (13) and (9) are transformed respectively to

$$a_1 t_1^n + b_1 t_1^l + c_1 = 0 \quad (14)$$

$$a_2 t_1^n + b_2 t_1^l + c_2 = 0 \quad (15)$$

where

$$a_1 = n^2 \left[(1-\eta_m)^n + R(1-\eta_m)^{2n-1} \right] - n \left[\frac{1}{R} + (1-\eta_m)^n \right] \left[1 + nR(1-\eta_m)^{n-1} \right] \quad (16)$$

$$\begin{aligned} b_1 = -nSl(1-\eta_m)^{n-1} \left[\eta_m + (1-\eta_m)^{l+1} - (1-\eta_m)^l \right] - nS \left[\frac{1}{R} + (1-\eta_m)^n \right] \left[1 - (l+1)(1-\eta_m)^l \right. \\ \left. + l(1-\eta_m)^{l-1} \right] \end{aligned} \quad (17)$$

$$c_1 = n \left[\frac{1}{R} + (1-\eta_m)^n \right] \quad (18)$$

$$a_2 = (1-\eta_m) + R(1-\eta_m)^n \quad (19)$$

$$b_2 = -S \left[\eta_m + (1-\eta_m)^{l+1} - (1-\eta_m)^l \right] \quad (20)$$

$$c_2 = -[(1-\eta_m) + R\tau^n] \quad (21)$$

The solution to the above system of equations can easily be obtained by simple elimination. Performing this step, t_1 is found to be given by

$$t_1 = \left[\frac{a_1 c_2 - a_2 c_1}{a_2 b_1 - a_1 b_2} \right]^{1/l} = \left[\frac{c_2 b_1 - c_1 b_2}{a_1 b_2 - a_2 b_1} \right]^{1/n} \quad (22)$$

and the efficiency at maximum power is obtained from the following equation:

$$\left[(1 - \eta_m) + R(1 - \eta_m)^n \right] \left[\frac{a_1 c_2 - a_2 c_1}{a_2 b_1 - a_1 b_2} \right]^{n/l} + S \left[\eta_m + (1 - \eta_m)^{l+1} - (1 - \eta_m)^l \right] \left[\frac{a_1 c_2 - a_2 c_1}{a_2 b_1 - a_1 b_2} \right] - (1 - \eta_m) - R\tau^n = 0 \tag{23}$$

The above equation may be solved numerically to obtain the value of the efficiency at maximum power once the values of the respective constants have been fixed. The numerical solution may easily be obtained by using the converging bisection method since the solution is known to be in [0,1].

Case 1: $n = m = 1, l = 4$. Equation (23) is solved numerically in this section, using the bisection method, for the case when $n = m = 1$ and $l = 4$. This is equivalent to assuming that heat transfer between the engine and the hot and cold reservoirs takes place via conduction and/or convection mechanism, while leakage occurs through radiation. A possible example of such situation could be a thermionic diode with conductive/convective heat input/output and radiative leakage from the cathode to the anode.

Results generated are depicted in Figures 2, 3 and 4 where the efficiency at maximum power (η_m) is plotted as a function of R (the ratio of conduction/convection heat transfer coefficients) for different values of τ (the ratio of cold to hot reservoir temperatures) at $S = 0.1, 0.5, \text{ and } 1$ respectively. The trends in the three figures are similar and show η_m to fall with increasing S (i.e., losses) as expected. Furthermore, for $\tau \neq 0$, results indicate a limited range for R (i.e., $0 \leq R \leq R_c$, R_c being the critical value of R beyond which physical solutions are not possible) within which physical solutions are possible (i.e., solutions for which $\eta_m \geq 0$ and $W \geq 0$) with the width of this range decreasing with increasing τ values for a given S . This can be explained by noticing that, as τ increases, the temperatures of the hot and cold reservoirs become closer. Since the same power law governs heat transfer through the hot and cold reservoirs and since heat rejected can be at most equal to heat absorbed, the range over which the heat transfer coefficients, and hence R , can vary decreases. The same conclusion may be obtained by solving equation (10) for the value of R at which the work becomes zero. In addition, the values of R at which

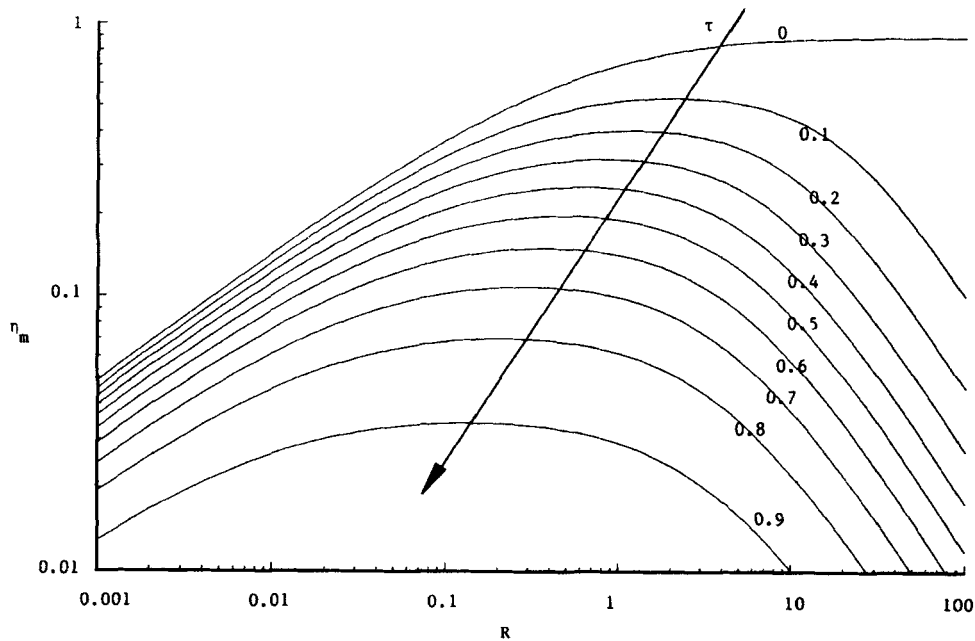


Figure 2. Variation of the efficiency at maximum power with R for different values of τ ($S = 0.1, n = m = 1, l = 4$)

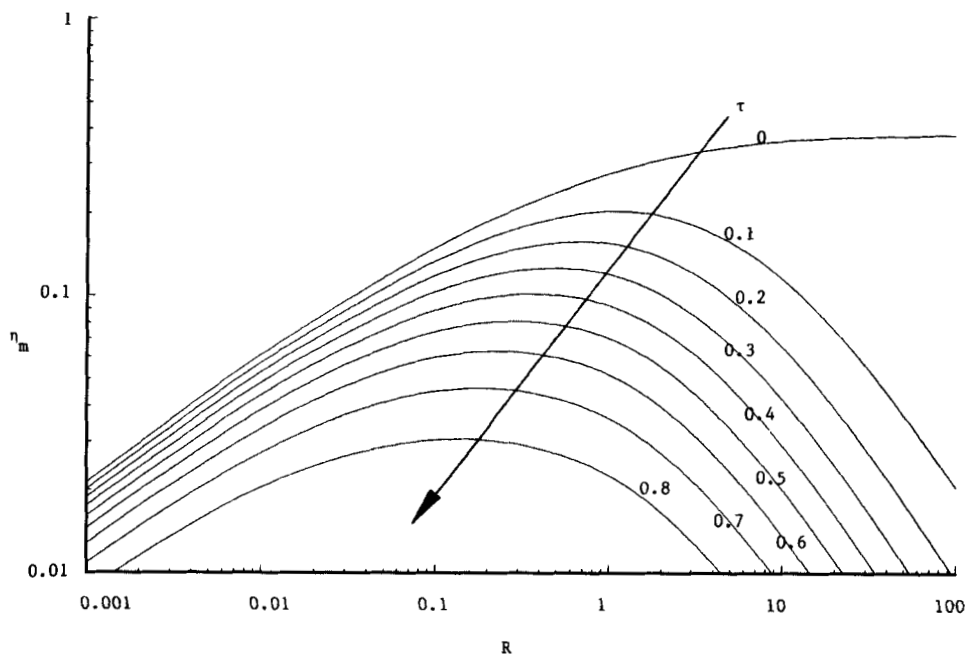


Figure 3. Variation of the efficiency at maximum power with R for different values of τ ($S = 0.5, m = m = 1, l = 4$)

the efficiency curves at maximum power ($\tau \neq 0$) peak, decrease with decreasing τ for the above-stated reasons.

For the case when $\tau = 0$, the range of physical solutions extends to infinity because the heat transfer coefficient β , and consequently R , can assume any value since it is multiplied by a zero temperature. In

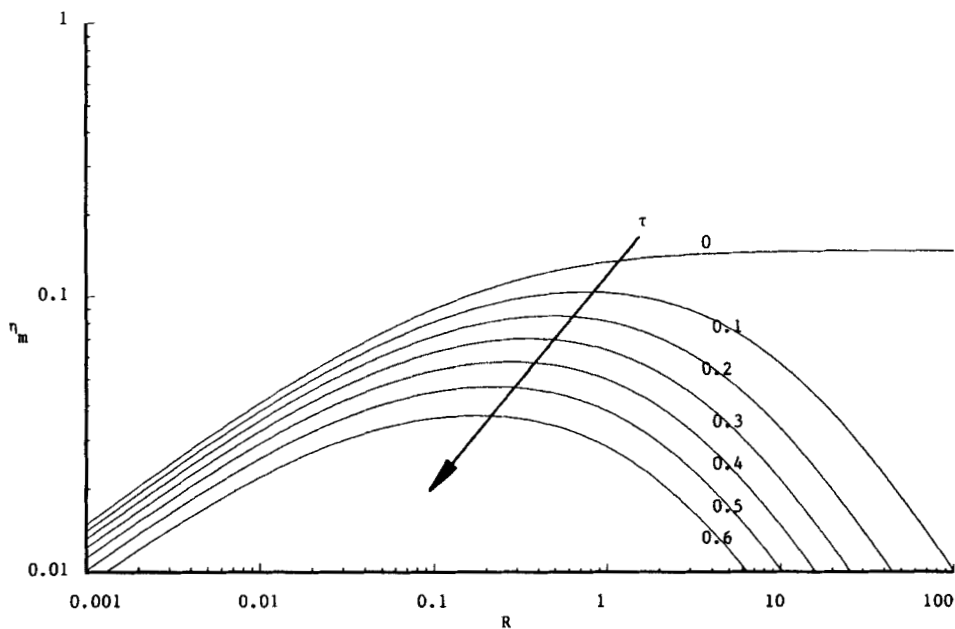


Figure 4. Variation of the efficiency at maximum power with R for different values of τ ($S = 1, n = m = 1, l = 4$)

addition, η_m is seen to increase asymptotically reaching a limiting value. This value decreases (from 1 when $S = 0$) as S increases.

Case 2: $n = m = 4, l = 1$. The exponents $n, m,$ and l are substituted by their respective values in equation (23) and the resulting equation is solved numerically to find the efficiency at maximum power for the situation in which heat transfer between the engine and the hot and cold reservoirs is taking place by radiation ($n = m = 4$) while leakage is occurring through the conductive/convective modes of heat transfer ($l = 1$). A first application of this situation is the calculation of solar-to-wind energy conversion introduced by DeVos (1991) and further studied by Nuwayhid and Moukalled (1994). A second example could be a thermionic diode with radiative heat input/output and conductive heat leak through the external leads. A thermoelectric generator with radiative input/output heat transfer mode and conductive/convective-mode of heat leak could be considered as a third possibility.

Figures 5, 6 and 7 show η_m versus R and τ for $S = 0.1, S = 0.5,$ and $S = 1$ respectively. The general trends of results are similar to those found in case 1 with the maximum achievable η_m (i.e., that for $\tau = 0$) dropping as S increases. Furthermore, an interesting observation to be made here is the insensitivity of η_m to variations in R over a wide range, especially at low values of τ as shown in Figure 5, with this insensitivity decreasing with S (Figures 6 and 7). This behaviour is due to the high powers involved which require larger changes in R (or β) to cause noticeable change in the heat rejected and consequently in the efficiency. This effect of the heat transfer power law, of higher influence at smaller values of $\tau,$ should clearly decrease with increasing values of τ or T_2 .

The general case $n = m = l$

Another special case of interest is the case for which $n = m = l$ whereby the same mode of heat transfer governs all heat transfer processes. In this situation, the solution to the above system of equations gives the following explicit relations for t_1 and η at maximum power (η_m):

$$t_1 = \left[-\frac{a_1}{b_1} \right]^{\frac{1}{n}} \tag{24}$$

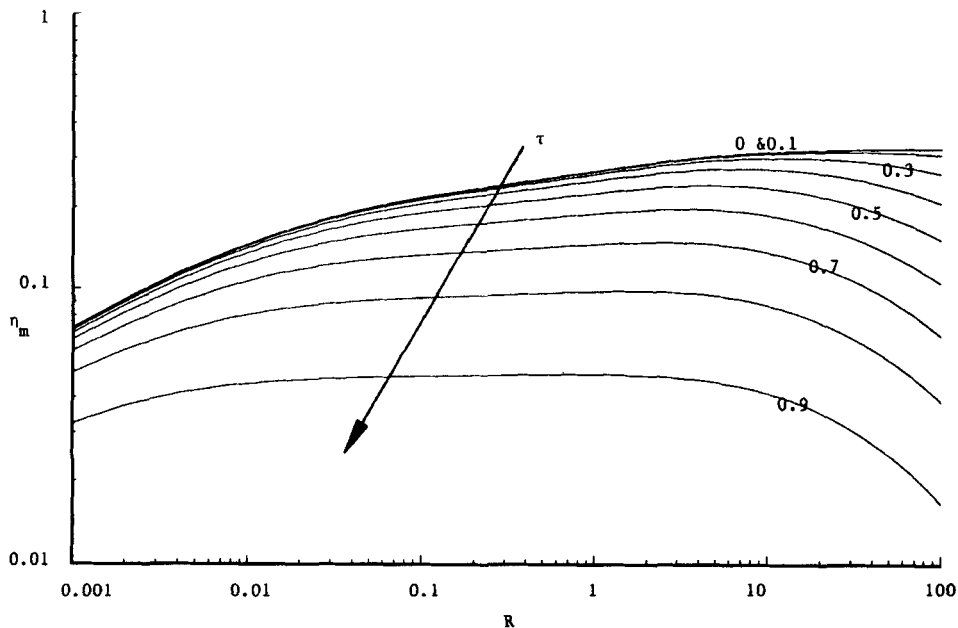


Figure 5. Variation of the efficiency at maximum power with R for different values of τ ($S = 0.1, n = m = 4, l = 1$)

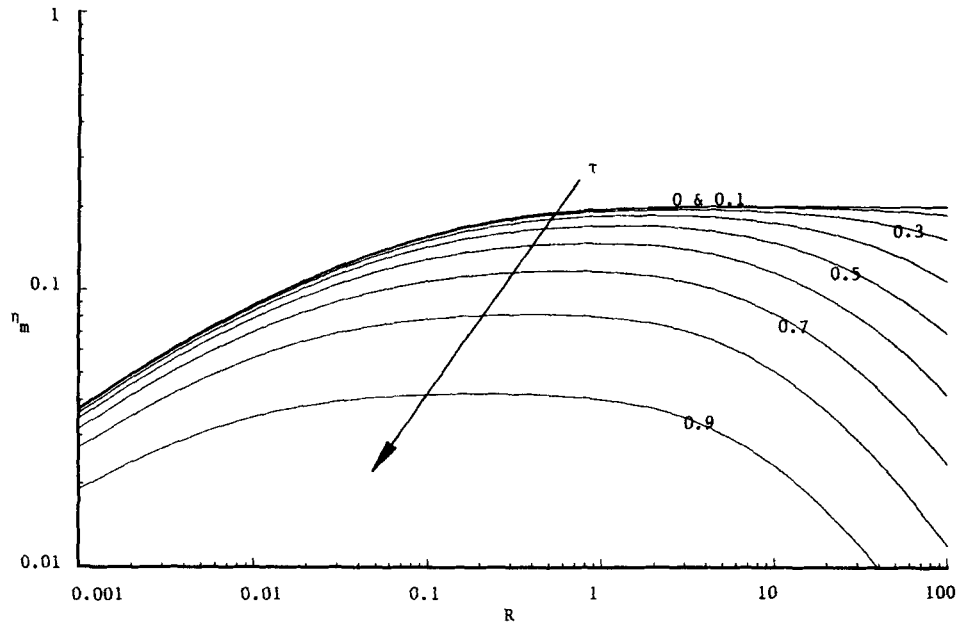


Figure 6. Variation of the efficiency at maximum power with R for different values of τ ($S = 0.5, n = m = 4, l = 1$)

where

$$a_1 = n \left[\frac{1}{R} + (1 - \eta_m)^n \right] \tag{25}$$

$$b_1 = n^2 \left[(1 - \eta_m)^n + R(1 - \eta_m)^{2n-1} \right] - n \left[\frac{1}{R} + (1 - \eta_m)^n \right] \left[1 + nR(1 - \eta_m)^{n-1} \right] \\ + n^2 S(1 - \eta_m)^{n-1} \left[-1 + (1 - \eta_m) - (1 - \eta_m)^{n+1} + (1 - \eta_m)^n \right] \\ - nS \left[\frac{1}{R} + (1 - \eta_m)^n \right] \left[1 - (n + 1)(1 - \eta_m)^n + n(1 - \eta_m)^{n-1} \right] \tag{26}$$

and

$$R[R + S(1 + \tau^n R)](1 - \eta_m)^{2n} + n(S + R + RS)(1 - \eta_m)^{n+1} + \{(1 - n)[(R + S) - R^2 \tau^n (1 + S)] \\ - RS(1 + n)(1 - \tau^n)\}(1 - \eta_m)^n - n\tau^n R[R + RS + S](1 - \eta_m)^{n-1} \\ - (S + R\tau^n + RS\tau^n) = 0 \tag{27}$$

Having derived the general equation that η_m should satisfy, the exponent n in equation (27) is assigned, consecutively, the values $-1, 1,$ and 4 . These chosen exponents, describe the well-known Fourier, Newton, and radiative heat transfer laws. As shown next, many established formulae derived in several references are easily obtained here as a very special case of the general formulation.

Case 1: $n = m = l = -1$. The heat transfer law governing all heat exchange processes is Fourier's law used in irreversible thermodynamics. Under these conditions, equation (27) reduces to

$$[R(1 + \tau) + S(2\tau + R + \tau R)](1 - \eta_m)^2 - 2[R(\tau - R) + S(\tau - R^2)](1 - \eta_m) \\ - R^2(1 + \tau) - SR(2R + 1 + \tau) = 0 \tag{28}$$

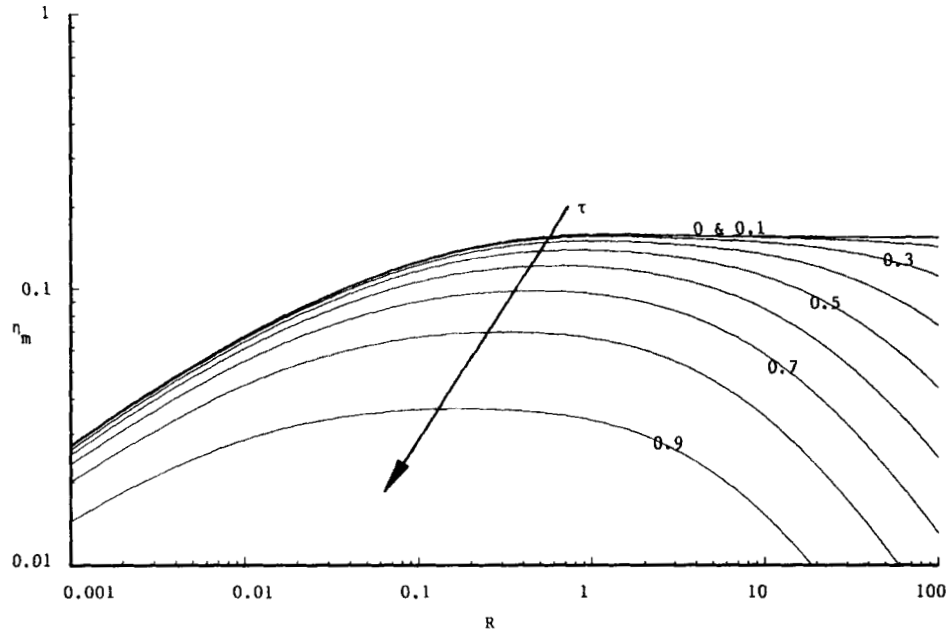


Figure 7. Variation of the efficiency at maximum power with R for different values of τ ($S = 1, n = m = 4, l = 1$)

This quadratic equation has the general solution,

$$\eta_m = 1 - \frac{[R(\tau - R) + S(\tau - R^2)] + \sqrt{[R(\tau - R) + S(\tau - R^2)]^2 + R[R(1 + \tau) + S(2\tau + R + \tau R)][R(1 + \tau) + S(1 + \tau + 2R)]}}{R(1 + \tau) + S(2\tau + R + \tau R)} \quad (29)$$

for given values of $R, S,$ and τ . If there is no heat leak, $S = 0$, then,

$$\eta_m = 1 - \frac{\tau - R}{\tau + 1} - \sqrt{\left(\frac{\tau - R}{\tau + 1}\right)^2 + R} \quad (30)$$

If, further, $R = 1$ indicating similar heat transfer coefficient into and out of the engine, the following relation for the efficiency at maximum power is obtained:

$$\lim_{R \rightarrow 1} \eta_m = \frac{2 - \sqrt{2(1 + \tau^2)}}{1 + \tau} \quad (31)$$

On the other hand, if $S = 0$ and $R = \infty$, then

$$\lim_{R \rightarrow \infty} \eta_m = \frac{1}{2}(1 - \tau) \quad (32)$$

This is the result obtained by Chen and Yan (1989) and DeVos (1985) when the only thermal resistance is between the working fluid and the high temperature source. Additionally, if $S = 0$ and $R = 0$ the efficiency at maximum power, as found by Chen and Yan (1989), is given by

$$\lim_{R \rightarrow 0} \eta_m = \frac{1 - \tau}{1 + \tau} \quad (33)$$

The above equation is valid when the only thermal resistance is between the working fluid and the low temperature source.

Case 2: $n = m = l = 1$. Substituting $n = 1$ into equations (1) and (27), then equation (1) expresses Newton's law and the relations for t_1 , W_m , and η_m reduce respectively to

$$t_1 = \frac{1 - \eta_m + \tau R}{(1 - \eta_m)(1 + R) - \eta_m^2 S} \quad (34)$$

$$\frac{W_m}{\alpha T_1} = 1 + \tau R - \frac{[1 + R(1 - \eta_m)][(1 - \eta_m) + \tau R]}{[(1 - \eta_m)(1 + R) - \eta_m^2 S]} \quad (35)$$

and

$$[R^2(1 + S\tau) + R(1 + 2S) + S]\eta_m^2 - 2[R^2(1 + S\tau) + R(1 + S + S\tau) + S]\eta_m + R(1 + R)(1 - \tau) = 0 \quad (36)$$

The above equation for η_m has always the following closed form solution:

$$\eta_m = 1 - \frac{(1 - \tau)RS}{R^2(1 + \tau S) + R(1 + 2S) + S} - \sqrt{\left[1 - \frac{(1 - \tau)RS}{R^2(1 + \tau S) + R(1 + 2S) + S}\right]^2 - \frac{R(1 + R)(1 - \tau)}{R^2(1 + \tau S) + R(1 + 2S) + S}} \quad (37)$$

It is interesting to note that when there is no leakage ($S = 0$), equation (37) above, irrespective of the value of R , reduces to

$$\eta_m = 1 - \sqrt{\tau} \quad (38)$$

which is the Curzon–Ahlborn efficiency (1975). Thus, equation (37) is the general form of that efficiency.

The maximum work under these conditions is obtained from the following simple equation:

$$\frac{W_m}{\alpha T_1} = \frac{R}{R + 1} \eta_m^2 \quad (39)$$

In addition, equation (37), for $\tau = 0$ (the case of maximum efficiency at maximum power) and with $R = S$ reduces to

$$\eta_m = 1 - \frac{R + \sqrt{(R + 1)(R + 2)}}{3R + 2} \quad (40)$$

The variation of η_m with R , as described by equation (40), is very slow and is confined to a range of lower limit 0.292 (as $R \rightarrow 0$) and of upper limit 0.333 (as $R \rightarrow \infty$). Furthermore, for $R = S = 1$, the highest η_m is 0.31. Nevertheless, the highest possible efficiency as shown by equation (38) is when $S = 0$. Thus, leakage governs the highest achievable efficiency at maximum power.

With $R = S = 1$, the efficiency at maximum power can be written as a function of τ as follows:

$$\eta_m = 1 - \frac{1 - \tau}{5 + \tau} - \sqrt{1 - \frac{4(1 - \tau)}{5 + \tau} + \left(\frac{1 - \tau}{5 + \tau}\right)^2} \quad (41)$$

For $\tau = 0$ the above equation gives $\eta = 0.31$ as before.

Case 3: $n = m = l = 4$. For this case, all heat transfer processes including the heat leak, take place through a radiative heat transfer mode. Upon inserting the respective powers into equation (24), the following relations for t_1 , W_m , and η_m are obtained:

$$t_1 = \left[\frac{(1 - \eta_m) + \tau^4 R}{(R + \eta_m S)(1 - \eta)^4 + (1 - \eta_m) - \eta_m S} \right]^{1/4} \quad (42)$$

$$\frac{W_m}{\alpha T_1^4} = 1 + R\tau^4 - \frac{[(1 - \eta_m) + \tau^4 R][1 + R(1 - \eta_m)^4]}{[(R + \eta_m S)(1 - \eta_m)^4 + (1 - \eta_m) - \eta_m S]} \quad (43)$$

and

$$\begin{aligned}
 &R[S + R + \tau^4 SR](1 - \eta_m)^8 + 4[RS + R + S](1 - \eta_m)^5 \\
 &\quad - [3S + 5RS + 3R - 3SR^2\tau^4 - 5RS\tau^4 - 3R^2\tau^4](1 - \eta_m)^4 \\
 &\quad - 4R[RS + R + S]\tau^4(1 - \eta_m)^3 - S - R\tau^4 - RS\tau^4 = 0
 \end{aligned} \tag{44}$$

When there is no leakage ($S = 0$), the equation reduces to

$$R(1 - \eta_m)^8 + 4(1 - \eta_m)^5 + 3(R\tau^4 - 1)(1 - \eta_m)^4 - 4R\tau^4(1 - \eta_m)^3 - \tau^4 = 0 \tag{45}$$

If, further to $S = 0$, $\tau = 0$ then we arrive at the highest possible efficiency at maximum power:

$$R(1 - \eta_m)^4 + 4(1 - \eta_m) - 3 = 0 \tag{46}$$

This equation reduces to that of DeVos (1991) when R approaches 1, i.e.,

$$\eta_m^4 - 4\eta_m^3 + 6\eta_m^2 - 8\eta_m + 2 = 0 \tag{47}$$

and the solution gives $\eta_m = 0.307$. Equation (46) is, however, the more general one in that the variation of efficiency at maximum power with R is shown.

When R approaches zero (while $S = 0$), it is seen from equation (46) that η_m approaches the value of 0.25, which appears to be a lower limit on the maximum efficiency at maximum power with no leakage. In general, using equation (27), it can easily be shown that such a limiting value exists (when $R \rightarrow 0$) for all heat transfer power modes as long as $n = m$. In fact, this limit turns out to be given by

$$\eta_m = \frac{1}{n} \tag{48}$$

Additionally, the normalized work as a function of R for $\tau = 0$ is:

$$\frac{W_m}{\alpha T_1^4} = 1 - \frac{(1 - \eta_m)[1 + R(1 - \eta_m)^4]}{(R + \eta_m S)(1 - \eta_m)^4 + (1 - \eta_m) - \eta_m S} \tag{49}$$

Furthermore, if $S = 0$ then the previous equation reduces to

$$\frac{W_m}{\alpha T_1^4} = 1 - \frac{1 + R(1 - \eta_m)^4}{1 + R(1 - \eta_m)^3} \tag{50}$$

For the case of $R = 1$ (and $S = 0$), this yields 0.0767 as reported in DeVos (1991). However, if R is altered the power output changes: at $R = 0.1$, $W_m/\alpha T_1^4 = 0.01012$; at $R = 1$, $W_m/\alpha T_1^4 = 0.0767$; at $R = 5$, $W_m/\alpha T_1^4 = 0.2078$; and at $R = 10$, $W_m/\alpha T_1^4 = 0.2813$. Thus, if the heat exhaust and input coefficients are different, the maximum extractable power changes. This implies for solar-to-wind energy conversion that 7.67% (even with no heat leak) is by no means a necessary upper limit.

As a common occurrence, if $R = S = 1$, the efficiency at maximum power turns out to be 0.08316 and the normalized work becomes $W = 0.02149$. Heat leak is thus shown to reduce the maximum conversion of solar energy into wind energy to a figure closer to the observed figures (DeVos (1991) and Gustavson (1979)). In general, however, the efficiency at maximum power in such a situation is a function of the leakage (S) and the ratio (R) as given by equations (44).

Finally, of interest is the case for which $S = 0$ and $R \rightarrow \infty$. Substitution of these values into equations (42)–(44) results in the following equations for the efficiency at maximum power, maximum power, and t_{01} :

$$\eta_m = 1 - \frac{T_2}{t_{01}} \tag{51}$$

$$W_m = \alpha \eta \left[T_1^4 - \frac{T_2^4}{(1 - \eta_m)^4} \right] \tag{52}$$

and

$$4 t_{01}^5 - 3 t_{01}^4 T_2 - T_1^4 T_2 = 0 \quad (53)$$

The last equation, known as the Castans relation (Chen and Yan, 1989), is a practical formula in solar energy conversion systems and shows again that the results of this paper are the most general.

CONCLUSION

The performance of endoreversible heat engines with heat leak was thoroughly investigated. A general formulation and a new solution methodology for the efficiency at maximum power were presented. Results have shown the efficiency at maximum power to be limited by the heat-leak mechanism and the predicted values were closer to the observed ones. Several new equations were presented and many well-known ones, developed in several references, were found to represent very special cases of the general formulation. The model can be applied to predicting the performance of a variety of energy conversion systems of practical interest.

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NOMENCLATURE

a_1, a_2	= coefficients in algebraic equations
b_1, b_2	= coefficients in algebraic equations
c_1, c_2	= coefficients in algebraic equations
l	= integer representing the heat transfer mode
m	= integer representing the heat transfer mode
n	= integer representing the heat transfer mode
Q_1	= heat entering the endoreversible heat engine
Q_2	= heat leaving the endoreversible heat engine
R	= dimensionless parameter
R_c	= critical value of R
S	= dimensionless parameter
t_1	= dimensionless temperature of the working material entering the engine
t_{01}	= dimensional temperature of the working material entering the engine
t_{02}	= dimensional temperature of the working material leaving the engine
T_1	= dimensional temperature of hot reservoir
T_2	= dimensional temperature of cold reservoir
W	= power output
W_m	= maximum power output
α, β, γ	= heat transfer coefficients
η	= efficiency
η_m	= efficiency at maximum power
τ	= ratio of cold to hot reservoir temperatures

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